CS 441 Discrete Mathematics for CS
Lecture 21

Mathematical induction & Recursion

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Course administration

- **Homework 7 is out**
  - **It is due on March 17, 2006**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Mathematical induction

- Used to prove statements of the form $\forall x\ P(x)$ where $x \in Z^+$

Mathematical induction proofs consists of two steps:

1) **Basis:** The proposition P(1) is true.
2) **Inductive Step:** The implication
   $P(n) \rightarrow P(n+1)$, is true for all positive n.

- Therefore we conclude $\forall x\ P(x)$.

Mathematical induction

Example: Prove the sum of first n odd integers is n^2 .
i.e. $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$ for all positive integers.

Proof:

- What is P(n)? P(n): $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$

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Basic Step

Mathematical induction

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Basis Step Show P(1) is true

- Trivial: $1 = 1^2$

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- Suppose P(n) be true, that is $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$
- Show P(n+1): $1 + 3 + 5 + 7 + ... + (2n - 1) + (2n + 1) =(n+1)^2$ follows:
- $\underbrace{1 + 3 + 5 + 7 + ... + (2n - 1)}_{n^2} + (2n + 1) =$

  $n^2 + (2n+1) = (n+1)^2$

Mathematical induction

Example: Prove $n^3 - n$ is divisible by 3 for all positive integers.

- P(n): $n^3 - n$ is divisible by 3

Basis Step: P(1): $1^3 - 1 = 0$ is divisible by 3 (obvious)

Mathematical induction

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- Suppose P(n): $n^3 - n$ is divisible by 3 is true.
- Show P(n+1): $(n+1)^3 - (n+1)$ is divisible by 3.

Mathematical induction

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- Show P(n+1): $(n+1)^3 - (n+1)$ is divisible by 3.

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1$$
$$= (n^3 - n) + 3n^2 + 3n$$
$$= \underbrace{(n^3 - n)}_{\text{divisible by 3}} + \underbrace{3(n^2 + n)}_{\text{divisible by 3}}$$

Strong induction

- **The regular induction:**
  - uses the basic step **P(1)** and
  - inductive step **P(n-1) → P(n)**
- **Strong induction uses:**
  - Uses the basis step **P(1)** and
  - inductive step **P(1) and P(2) … P(n-1) → P(n)**

Example: Show that a positive integer greater than 1 can be written as a product of primes.

Strong induction

Example: Show that a positive integer greater than 1 can be written as a product of primes.

Assume P(n): an integer n can be written as a product of primes.

Basis step: P(2) is true

Inductive step: Assume true for P(2),P(3), … P(n)

Show that P(n+1) is true as well.

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2 Cases:

- If n+1 is a prime then P(n+1) is trivially true

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- If n+1 is a prime then P(n+1) is trivially true
- If n+1 is a composite then it can be written as a product of two integers $(n+1) = a*b$ such that $1< a ,b < n+1$

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Show that P(n+1) is true as well.

2 Cases:

- If n+1 is a prime then P(n+1) is trivially true
- If n+1 is a composite then it can be written as a product of two integers $(n+1) = a*b$ such that $1< a ,b < n+1$
- From the assumption P(a) and P(b) holds.
- Thus, n+1 can be written as a product of primes
- **End of proof**

Recursive Definitions

- Sometimes it is difficult to define an object explicitly, however it may be easy to define the object in terms of itself. This process is called **recursion.**

Examples of recursive definitions:

- Recursive definition of a geometric sequence:
  - $x_n = 3^n$
  - $x_0 = 1;\ x_n = 3x_{n-1}$
- Algorithm for computing the gcd:
  - **gcd(79, 35) = gcd(35, 9)**

Recursively Defined Functions

To define a function on the set of nonnegative integers

- 1. Specify the value of the function at 0
- 2. Give a rule for finding the function's value at n+1 in terms of the function's value at integers $i \leq n$.

- **Such a definition is called recursive or inductive.**

Recursively defined functions

Example: Assume a recursive function on positive integers:

- $f(0) = 3$
- $f(n+1) = 2f(n) + 3$

- **What is the value of f(0) ?**

Recursively defined functions

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- $f(1) = ?$

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- **What is the value of f(0) ?** 3
- $f(1) = 2f(0) + 3 = 2(3) + 3 = 6 + 3 = 9$
- $f(2) = ?$

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- $f(2) = f(1 + 1) = 2f(1) + 3 = 2(9) + 3 = 18 + 3 = 21$
- $f(3) = ?$

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- $f(3) = f(2 + 1) = 2f(2) + 3 = 2(21) = 42 + 3 = 45$
- $f(4) = ?$

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- $f(3) = f(2 + 1) = 2f(2) + 3 = 2(21) = 42 + 3 = 45$
- $f(4) = f(3 + 1) = 2f(3) + 3 = 2(45) + 3 = 90 + 3 = 93$

Recursive definitions

- **Example**

  Define the function:

  $f(n) = 2n + 1 \quad n = 0, 1, 2, \ldots$

  recursively.

- $f(0) = ?$

Recursive defined functions

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  Define the function:

  $f(n) = 2n + 1 \quad n = 0, 1, 2, \ldots$

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- $f(0) = 1$
- $f(n+1) = ?$

Recursive definitions

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  Define the function:

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  recursively.

- $f(0) = 1$
- $f(n+1) = f(n) + 2$

Recursive definitions

- **Example:**

  Define the sequence:

  $a_n = n^2 \quad$ for $n = 1,2,3, \ldots$

  recursively.

- $a_1 = ?$

Recursive definitions

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  Define the sequence:
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Recursive definitions

- **Example:**
  Define the sequence:
  $a_n = n^2$ for $n = 1,2,3, \ldots$
  recursively.

- $a_1 = 1$
- $a_{n+1} = a_n^2 + (2n + 1), \quad n \geq 1$

Recursive definitions

- **Example:**
  Define a recursive definition of the sum of the first n positive integers:

$$F(n) = \sum_{i=1}^{n} i$$

- F(1) = ?

Recursive definitions

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  Define a recursive definition of the sum of the first n positive integers:

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- F(1) = 1
- F(n+1) =?

Recursive definitions

- **Example:**
  Define a recursive definition of the sum of the first n positive integers:

$$F(n) = \sum_{i=1}^{n} i$$

- $F(1) = 1$
- $F(n+1) = F(n) + (n+1)$, $n \geq 1$

Recursive definitions

Some important functions or sequences in mathematics are defined recursively

Factorials

- $n! = 1$ if $n=1$
- $n! = n.(n-1)!$ if $n \geq 1$

Fibonacci numbers:

- $F(0)=0$, $F(1)=1$ and
- $F(n)=F(n-1) + F(n-2)$ for $n=2,3, \ldots$

Recursive definitions

- **Greatest common divisor**

  gcd(a,b) = b if b | a

  = gcd(b, a mod b)

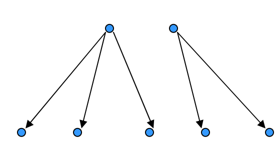
Recursive definitions

- **Data structures**

Example: Rooted tree

- **A basis step:**
  – a single node (vertex) is a rooted tree

- **Recursive step:**
  – Assume T1, T2, … Tk are rooted trees, then the graph with a root r connected to T1, T2, … Tk is a rooted tree



Recursive definitions

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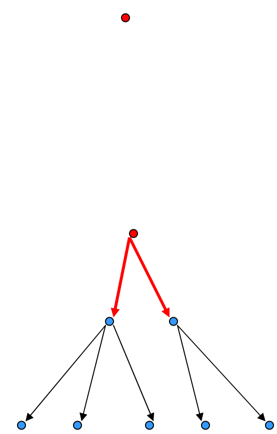

Recursive definitions

- **Assume the alphabet Σ**
  – Example: $\Sigma = \{a,b,c,d\}$
- **A set of all strings containing symbols in Σ: Σ^***
  – Example: $\Sigma^* = \{$“”,a,aa,aaa,aaa…, ab, …b,bb, bbb, …$\}$

Recursive definition of Σ^*

- **Basis step:**
  – empty string $\lambda \in \Sigma^*$
- **Recursive step:**
  – If $w \in \Sigma^*$ and $x \in \Sigma$ then $wx \in \Sigma^*$