Mathematical induction & Recursion

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Course administration

- Homework 7 is out
  - It is due on March 17, 2006

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/
Mathematical induction

- Used to prove statements of the form $\forall x \ P(x)$ where $x \in \mathbb{Z}^+$

Mathematical induction proofs consists of two steps:

1) **Basis:** The proposition $P(1)$ is true.
2) **Inductive Step:** The implication $P(n) \rightarrow P(n+1)$, is true for all positive $n$.

- Therefore we conclude $\forall x \ P(x)$.

**Example:** Prove the sum of first $n$ odd integers is $n^2$.

i.e. $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$ for all positive integers.

**Proof:**

- What is $P(n)$? $P(n): 1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$
Mathematical induction

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Proof:
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Basic Step

Basis Step
• Show $P(1)$ is true
• Trivial: $1 = 1^2$
**Mathematical induction**

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**Proof:**

- What is $P(n)$? $P(n): 1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$

**Basis Step** Show $P(1)$ is true
- Trivial: $1 = 1^2$

**Inductive Step** Show if $P(n)$ is true then $P(n+1)$ is true for all $n$.

- Suppose $P(n)$ be true, that is $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$
- Show $P(n+1)$: $1 + 3 + 5 + 7 + ... + (2n - 1) + (2n + 1) = (n+1)^2$

follows:

\[
\frac{1 + 3 + 5 + 7 + ... + (2n - 1) + (2n + 1)}{n^2 + (2n+1)} = (n+1)^2
\]
Mathematical induction

Example: Prove \( n^3 - n \) is divisible by 3 for all positive integers.

- **P(n):** \( n^3 - n \) is divisible by 3

**Basis Step:** \( P(1): \quad 1^3 - 1 = 0 \) is divisible by 3 (obvious)

**Inductive Step:** If \( P(n) \) is true then \( P(n+1) \) is true for each positive integer.

- Suppose \( P(n): \quad n^3 - n \) is divisible by 3 is true.
- Show \( P(n+1): \quad (n+1)^3 - (n+1) \) is divisible by 3.
**Mathematical induction**

**Example:** Prove $n^3 - n$ is divisible by 3 for all positive integers.
- **P(n):** $n^3 - n$ is divisible by 3

**Basis Step:** $P(1): 1^3 - 1 = 0$ is divisible by 3 (obvious)

**Inductive Step:** If $P(n)$ is true then $P(n+1)$ is true for each positive integer.
- Suppose $P(n): \ n^3 - n$ is divisible by 3 is true.
- Show $P(n+1): (n+1)^3 - (n+1)$ is divisible by 3.

\[
(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 \\
= (n^3 - n) + 3n^2 + 3n \\
= (n^3 - n) + 3(n^2 + n) \\
\text{divisible by 3} \quad \text{divisible by 3}
\]

**Strong induction**

- **The regular induction:**
  - uses the basic step $P(1)$ and
  - inductive step $P(n-1) \Rightarrow P(n)$
- **Strong induction uses:**
  - Uses the basis step $P(1)$ and
  - inductive step $P(1)$ and $P(2)$ … $P(n-1) \Rightarrow P(n)$

**Example:** Show that a positive integer greater than 1 can be written as a product of primes.
Strong induction

**Example:** Show that a positive integer greater than 1 can be written as a product of primes.
Assume P(n): an integer n can be written as a product of primes.
**Basis step:** P(2) is true
**Inductive step:** Assume true for P(2), P(3), … P(n)
Show that P(n+1) is true as well.

2 Cases:
• If n+1 is a prime then P(n+1) is trivially true
**Strong induction**

**Example:** Show that a positive integer greater than 1 can be written as a product of primes.

Assume P(n): an integer n can be written as a product of primes.

**Basis step:** P(2) is true

**Inductive step:** Assume true for P(2), P(3), … P(n)

Show that P(n+1) is true as well.

2 Cases:
- If n+1 is a prime then P(n+1) is trivially true
- If n+1 is a composite then it can be written as a product of two integers (n+1) = a*b such that 1 < a, b < n+1

End of proof
Recursive Definitions

• Sometimes it is difficult to define an object explicitly, however it may be easy to define the object in terms of itself. This process is called recursion.

Examples of recursive definitions:
• Recursive definition of a geometric sequence:
  • $x_n = 3^n$
  • $x_0 = 1; \ x_n = 3x_{n-1}$
• Algorithm for computing the gcd:
  • $\text{gcd}(79, 35) = \text{gcd}(35, 9)$

Recursively Defined Functions

To define a function on the set of nonnegative integers
• 1. Specify the value of the function at 0
• 2. Give a rule for finding the function's value at $n+1$ in terms of the function's value at integers $i \leq n$.

• Such a definition is called recursive or inductive.
Recursively defined functions

Example: Assume a recursive function on positive integers:
- \( f(0) = 3 \)
- \( f(n+1) = 2f(n) + 3 \)

- What is the value of \( f(0) \)?
- \( f(1) = ? \)
Recursively defined functions

Example: Assume a recursive function on positive integers:

- \( f(0) = 3 \)
- \( f(n+1) = 2f(n) + 3 \)

- What is the value of \( f(0) \)? 3
- \( f(1) = 2f(0) + 3 = 2(3) + 3 = 6 + 3 = 9 \)
- \( f(2) = ? \)

- \( f(3) = ? \)
Recursively defined functions

Example: Assume a recursive function on positive integers:

- $f(0) = 3$
- $f(n+1) = 2f(n) + 3$

- What is the value of $f(0)$? 3
- $f(1) = 2f(0) + 3 = 2(3) + 3 = 6 + 3 = 9$
- $f(2) = f(1 + 1) = 2f(1) + 3 = 2(9) + 3 = 18 + 3 = 21$
- $f(3) = f(2 + 1) = 2f(2) + 3 = 2(21) = 42 + 3 = 45$
- $f(4) = f(3 + 1) = 2f(3) + 3 = 2(45) + 3 = 90 + 3 = 93$
Recursive definitions

• Example
  Define the function:
  \[ f(n) = 2n + 1 \quad n = 0, 1, 2, \ldots \]
  recursively.

• \( f(0) = ? \)

Recursive defined functions

• Example
  Define the function:
  \[ f(n) = 2n + 1 \quad n = 0, 1, 2, \ldots \]
  recursively.

• \( f(0) = 1 \)
• \( f(n+1) = ? \)
Recursive definitions

• **Example:**
  Define the function:
  \[ f(n) = 2n + 1 \quad n = 0, 1, 2, ... \]
  recursively.

  • \( f(0) = 1 \)
  • \( f(n+1) = f(n) + 2 \)

Recursive definitions

• **Example:**
  Define the sequence:
  \[ a_n = n^2 \quad \text{for } n = 1, 2, 3, ... \]
  recursively.

  • \( a_1 = 1 \)
Recursive definitions

• Example:
  Define the sequence:
  \[ a_n = n^2 \text{ for } n = 1, 2, 3, \ldots \]
  recursively.

• \( a_1 = 1 \)
• \( a_{n+1} = ? \)
Recursive definitions

• Example:
   Define a recursive definition of the sum of the first n positive integers:
   \[ F(n) = \sum_{i=1}^{n} i \]

   • F(1) = 1
   • F(n+1) =?
Recursive definitions

- **Example:**
  Define a recursive definition of the sum of the first \( n \) positive integers:

  \[
  F(n) = \sum_{i=1}^{n} i
  \]

  - \( F(1) = 1 \)
  - \( F(n+1) = F(n) + (n+1) \), \( n \geq 1 \)

Some important functions or sequences in mathematics are defined recursively

**Factorials**
- \( n! = 1 \) if \( n=1 \)
- \( n! = n.(n-1)! \) if \( n \geq 1 \)

**Fibonacci numbers:**
- \( F(0)=0, \ F(1)=1 \) and
- \( F(n) = F(n-1) + F(n-2) \) for \( n=2,3, \ldots \)
Recursive definitions

• Greatest common divisor
  \[
  \text{gcd}(a,b) = \begin{cases} 
  b & \text{if } b \mid a \\
  \text{gcd}(b, a \mod b) & \text{otherwise}
  \end{cases}
  \]

• Data structures

Example: Rooted tree

• A basis step:
  – a single node (vertex)
    is a rooted tree

• Recursive step:
  – Assume T_1, T_2, … T_k are rooted
trees, then the graph with a root
r connected to T_1, T_2, … T_k is
  a rooted tree
Recursive definitions

- **Data structures**
  - **Example:** Rooted tree
- **A basis step:**
  - a single node (vertex) is a rooted tree
- **Recursive step:**
  - Assume T1, T2, ... Tk are rooted trees, then the graph with a root r connected to T1, T2, ... Tk is a rooted tree

Recursive definitions

- **Assume the alphabet Σ**
  - Example: Σ = {a,b,c,d}
- **A set of all strings containing symbols in Σ:** Σ*
  - Example: Σ* = {"",a,aa,aaa,aaa..., ab, ...b,bb, bbb, ...}

Recursive definition of Σ*

- **Basis step:**
  - empty string λ∈Σ*
- **Recursive step:**
  - If w∈Σ* and x∈Σ then wx∈Σ*