CS 441 Discrete Mathematics for CS
Lecture 20

Mathematical induction

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Course administration

- **Homework 6 is out**
  Due on Friday, March 3, 2006 or earlier (TA office)

- **Homework 7 is out, due on March 17, 2006**

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/

Proofs

Basic proof methods:

- Direct, Indirect, Contradiction, By Cases, Equivalences

Proof of quantified statements:

- **There exists x with some property P(x).**
  - It is sufficient to find one element for which the property holds.
- **For all x some property P(x) holds.**
  - Proofs of 'For all x some property P(x) holds' must cover all x and can be harder.
- **Mathematical induction** is the technique that can be applied to prove the universal statements for sets of positive integers or their associated sequences.

Mathematical induction

- Used to prove statements of the form $\forall x\ P(x)$ where $x \in Z^+$

Mathematical induction proofs consists of two steps:

1) **Basis:** The proposition P(1) is true.
2) **Inductive Step:** The implication

   $P(n) \rightarrow P(n+1)$, is true for all positive n.

- Therefore we conclude $\forall x\ P(x)$.

- **Based on the well-ordering property:** Every nonempty set of nonnegative integers has a **least element**.

Mathematical induction

Example: Prove the sum of first n odd integers is n^2 .

i.e. $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$ for all positive integers.

Proof:

- What is P(n)? P(n): $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$

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Basic Step

Mathematical induction

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Basis Step Show P(1) is true

- Trivial: $1 = 1^2$

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Inductive Step Show if P(n) is true then P(n+1) is true for all n.

- Suppose P(n) be true, that is $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$
- Show P(n+1): $1 + 3 + 5 + 7 + ... + (2n - 1) + (2n + 1) = (n+1)^2$ follows:
- $\underbrace{1 + 3 + 5 + 7 + ... + (2n - 1)}_{n^2} + (2n + 1) =$

$$n^2 + (2n+1) = (n+1)^2$$

Correctness of the mathematical induction

Suppose **P(1) is true** and **P(n) $\rightarrow$ P(n+1) is true** for all positive integers n. Want to show $\forall$**x P(x).**

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By the Well-Ordering Property, S has a least member, say k. k > 1, since P(1) is true. This implies k - 1 > 0 and P(k-1) is true (since remember k is the smallest integer where P(k) is false).

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Now: P(k-1) $\rightarrow$ P(k) is true

thus, P(k) must be true (a contradiction).

- **Therefore $\forall$x P(x).**

Mathematical induction

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$$
\begin{aligned}
n + 1 &< 2^n + 1 \\
&< 2^n + 2^n \\
&= 2^n(1 + 1) \\
&= 2^n (2) \\
&= 2^{n+1}
\end{aligned}
$$

Mathematical induction

Example: Prove $n^3 - n$ is divisible by 3 for all positive integers.

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$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1$$

$$= (n^3 - n) + 3n^2 + 3n$$

$$= (n^3 - n) + 3(n^2 + n)$$

⇗ divisible by 3 ⇖ divisible by 3

Strong induction

- **The regular induction:**
  - uses the basic step **P(1)** and
  - inductive step **P(n-1) → P(n)**
- **Strong induction uses:**
  - Uses the basis step **P(1)** and
  - inductive step **P(1) and P(2) … P(n-1) → P(n)**

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Strong induction

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Basis step: P(2) is true

Inductive step: Assume true for P(2),P(3), … P(n)

Show that P(n+1) is true as well.

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- If n+1 is a prime then P(n+1) is trivially true

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2 Cases:

- If n+1 is a prime then P(n+1) is trivially true
- If n+1 is a composite then it can be written as a product of two integers (n+1) = a*b such that 1< a ,b < n+1
- From the assumption P(a) and P(b) holds.
- Thus, n+1 can be written as a product of primes
- **End of proof**