CS 441 Discrete Mathematics for CS
Lecture 2

Propositional logic

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Propositional logic: review

- **Propositional logic:** a formal language for making logical inferences
- **A proposition is** a statement that is either true or false.
- **A compound proposition** can be created from other propositions using logical connectives
- **The truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- **The truth table for a compound proposition:** table with entries (rows) for all possible combinations of truth values of elementary propositions.

Compound propositions

- More complex propositional statements can be build from the elementary statements using **logical connectives.**

- **Logical connectives:**
  - Negation
  - Conjunction
  - Disjunction
  - Exclusive or
  - Implication
  - Biconditional

Compound propositions

- Let p: 2 is a prime
  q: 6 is a prime
- Determine **the truth value** of the following statements:

  $\neg$ p:

  p $\wedge$ q :

  p $\wedge$ $\neg$q:

  p $\vee$ q :

  p $\oplus$ q:

  p $\rightarrow$ q:

  q $\rightarrow$ p:

Compound propositions

- Let p: 2 is a prime ….. **T**
  q: 6 is a prime ….. **F**
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  p $\wedge$ $\neg$q: **T**

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  p $\vee$ q : **T**
  p $\oplus$ q:
  p $\rightarrow$ q:
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  p $\wedge$ $\neg$q: **T**
  p $\vee$ q : **T**
  p $\oplus$ q: **T**
  p $\rightarrow$ q:
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  $p \rightarrow q$: **F**

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  $p \vee q$: **T**

  $p \oplus q$: **T**

  $p \rightarrow q$: **F**

  $q \rightarrow p$: **T**

Implication

- The **converse of p → q** is q → p.
- The **contrapositive of p → q** is ¬q → ¬p
- The **inverse** of .**p → q** is ¬p → ¬q

- **Examples:**
    - If it snows, the traffic moves slowly.
    - p: it snows   q: traffic moves slowly.
    - **p → q**
  - **The converse:**
    If the traffic moves slowly then it snows.
    - **q → p**

Implication

- The **contrapositive of p → q** is ¬q → ¬p
- The **inverse** of .**p → q** is ¬p → ¬q

- **Examples:**
    - If it snows, the traffic moves slowly.
  - **The contrapositive:**
    - If the traffic does not move slowly then it does not snow.
    - ¬q → ¬p
  - **The inverse:**
    - If does not snow the traffic moves quickly.
    - ¬p → ¬q

Biconditional

- **Definition**: Let p and q be propositions. The **biconditional** **p ↔ q** (read p if and only if q), is true when p and q have the same truth values and is false otherwise.

p	q	p ↔ q
T	T	
T	F	
F	T	
F	F	

- **Note:** two truth values always agree.

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p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

- **Note:** two truth values always agree.

Constructing the truth table

- Examples: Construct the truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

p	q	¬p	p → q	¬p ↔ q	(p→q)∧ (¬p↔q)
T	T				
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Constructing the truth table

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T	F	F	F	T	
F	T	T	T	T	
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Constructing the truth table

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p	q	$\neg p$	$p \rightarrow q$	$\neg p \leftrightarrow q$	$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Translation

- Logic helps us to define the meaning of statements: Mathematical or English statements.
- **How to translate an English sentence to the logic?**
- **Assume a sentence:**
  – If you are older than 13 or you are with your parents then you can attend a PG-13 movie.
- The whole sentence is a proposition. It is **True.**
- But this is not the best. We want to parse the sentence to elementary statements that are combined with connectives.

Translation

If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

Parse:

- **If (you are older than 13 or you are with your parents) then (you can attend a PG-13 movie)**
  - A= you are older than 13
  - B= you are with your parents
  - C=you can attend a PG-13 movie
- **Translation: $A \vee B \rightarrow C$**

- **Why do we want to do this?**
- **Inference: Assume I know that $A \vee B \rightarrow C$ is a correct statement and both A and B are true. Then we can conclude that C is true as well.**

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

Translation

- Assume two elementary statements:
  - p: you drive over 65 mph **and** q: you get a speeding ticket
- **Translate each of these sentences to logic**
  - you do not drive over 65 mph

Translation

- Assume two elementary statements:
  - p: you drive over 65 mph **and** q: you get a speeding ticket
- **Translate each of these sentences to logic**
  - you do not drive over 65 mph. $(\neg p)$
  - you drive over 65 mph, but you don't get a speeding ticket.

Translation

- Assume two elementary statements:
  - p: you drive over 65 mph **and** q: you get a speeding ticket

Translate each of these sentences to logic

- you do not drive over 65 mph. ($\neg p$)
- you drive over 65 mph, but you don't get a speeding ticket. ($p \wedge \neg q$)
- you will get a speeding ticket if you drive over 65 mph.

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  - you do not drive over 65 mph. ($\neg p$)
  - you drive over 65 mph, but you don't get a speeding ticket. ($p \wedge \neg q$)
  - you will get a speeding ticket if you drive over 65 mph. ($p \rightarrow q$)
  - if you do not drive over 65 mph then you will not get a speeding ticket

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  - you do not drive over 65 mph. $(\neg p)$
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  - you will get a speeding ticket if you drive over 65 mph. $(p \rightarrow q)$
  - if you do not drive over 65 mph then you will not get a speeding ticket. $(\neg p \rightarrow \neg q)$
  - driving over 65 mph is sufficient for getting a speeding ticket.

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  - driving over 65 mph is sufficient for getting a speeding ticket. $(p \rightarrow q)$
  - you get a speeding ticket, but you do not drive over 65 mph.

Translation

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  - you do not drive over 65 mph. ($\neg p$)
  - you drive over 65 mph, but you don't get a speeding ticket. ($p \wedge \neg q$)
  - you will get a speeding ticket if you drive over 65 mph. ($p \rightarrow q$)
  - if you do not drive over 65 mph then you will not get a speeding ticket.($\neg p \rightarrow \neg q$)
  - driving over 65 mph is sufficient for getting a speeding ticket. ($p \rightarrow q$)
  - you get a speeding ticket, but you do not drive over 65 mph. ($q \wedge \neg p$)
  -

Computer representation of True and False

- **We need to encode two values True and False:**
  - **use a bit**
  - a bit represents two possible values:
  - 0 (False) or 1(True)
- A variable that takes on values 0 or 1 is called a **Boolean variable**.
- **Definition**: A **bit string** is a sequence of zero or more bits. The **length** of this string is the number of bits in the string.

Bitwise operations

- T and F replaced with 1 and 0

p	q	$p \vee q$	$p \wedge q$
1	1	1	1
1	0	1	0
0	1	1	0
0	0	0	0

p	$\neg p$
1	0
0	1

Bitwise operations

- **Examples:**

$$\begin{array}{r} 1011\ 0011 \\ \vee\ \underline{0110\ 1010} \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \wedge\ \underline{0110\ 1010} \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \oplus\ \underline{0110\ 1010} \end{array}$$

Bitwise operations

- **Examples:**

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