CS 441 Discrete Mathematics for CS
Lecture 19

Summations, Cardinality

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Course administration

- **Homework 5 is due today**
- **Homework 6 is out**
  Due on Friday, March 3, 2006 or earlier (TA office)

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/

Arithmetic series

Definition: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is called an **arithmetic series**.

Theorem: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is

$$S = \sum_{j=1}^{n}(a + jd) = na + d\sum_{j=1}^{n} j = na + d\,\frac{n(n+1)}{2}$$

Geometric series

Definition: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^k is called **a geometric series**.

Theorem: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^n is

$$S = \sum_{j=0}^{n}(ar^j) = a\sum_{j=0}^{n} r^j = a\left[\frac{r^{n+1}-1}{r-1}\right]$$

Geometric series

Theorem: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^n is

$$S = \sum_{j=0}^{n} (ar^j) = a\sum_{j=0}^{n} r^j = a\left[\frac{r^{n+1}-1}{r-1}\right]$$

Proof:

$$S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + \ldots + ar^n$$

Infinite geometric series

- **Infinite geometric series can be computed in the closed form for x<1**

$$S = \sum_{n=0}^{\infty} (x^n)$$

- **How?**

Infinite geometric series

- **Infinite geometric series can be computed in the closed form for x<1**
- **How?**

$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} \sum_{n=0}^{k} x^n = \lim_{k\to\infty} \frac{x^{k+1}-1}{x-1} = -\frac{1}{x-1} = \frac{1}{1-x}$$

- **Thus:**

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Cardinality

Recall: The cardinality of a finite set is defined by the number of elements in the set.

Definition: The sets A and B have **the same cardinality** if there is a one-to-one correspondence between elements in A and B. In other words if there is a bijection from A to B. Recall bijection is one-to-one and onto.

Example: Assume A = {a,b,c} and B = {α,β,γ} and function f defined as:

- a → α
- b → β
- c → γ

F defines a bijection. Therefore A and B have the same cardinality, i.e. | A | = | B | = 3.

Cardinality

Definition: A set that is either finite or has the same cardinality as the set of positive integers Z^+ is called **countable.** A set that is not countable is called **uncountable.**

Why these are called countable? The elements of the set can be enumerated and listed.

Countable sets

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Z+ = {1, 2, 3, 4, …}

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- Define a function f: $x \rightarrow 2x - 2$ (an arithmetic progression)
  - $1 \rightarrow 2(1)-2 = 0$
  - $2 \rightarrow 2(2)-2 = 2$
  - $3 \rightarrow 2(3)-2 = 4$ ...

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  - $1 \rightarrow 2(1)-2 = 0$
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  - $3 \rightarrow 2(3)-2 = 4$ ...
- one-to-one (why?) $2x-2 = 2y-2 \Rightarrow 2x = 2y \Rightarrow x = y$.
- onto (why?) $\forall a \in A$, $(a+2) / 2$ is the pre-image in Z^+.
- Therefore $| A | = | Z^+ |$.

Cardinality

Theorem: The set of real numbers (R) is an uncountable set.

Proof by a contradiction.

1) Assume that the real numbers are countable.
2) Then every subset of the reals is countable, in particular, the interval from 0 to 1 is countable. This implies the elements of this set can be listed say r1, r2, r3, ... where

- $r1 = 0.d_{11}d_{12}d_{13}d_{14} \ldots$
- $r2 = 0.d_{21}d_{22}d_{23}d_{24} \ldots$
- $r3 = 0.d_{31}d_{32}d_{33}d_{34} \ldots\ldots$
- where the $d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$.

Real numbers are uncountable

Proof cont.

3) Want to show that not all reals in the interval between 0 and 1 are in this list.

- Form a new number called
  - $r = 0.d_1 d_2 d_3 d_4 \ldots$ where

$$d_i = \begin{cases} 2, & \text{if } d_{ii} \neq 2 \\ 3 & \text{if } d_{ii} = 2 \end{cases}$$

- Example: suppose

r1 = 0.**7**5243...	d1 = 2
r2 = 0.5**2**4310...	d2 = 3
r3 = 0.13**1**257...	d3 = 2
r4 = 0.936**3**633...	d4 = 2
...	...
rt = 0.23222**2**22...	dt = 3

Real numbers are uncountable

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$$d_i = \begin{cases} 2, & \text{if } d_{ii} \neq 2 \\ 3 & \text{if } d_{ii} = 2 \end{cases}$$

- **Claim:** r is different than each member in the list.
- Is each expansion unique? Yes, if we exclude an infinite string of 9s.
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- Example: $.0285\overline{0} = .0284\overline{9}$
- Therefore r and r_i differ in the i-th decimal place for all i.