CS 441 Discrete Mathematics for CS
Lecture 18

Summations

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Course administration

- **Homework 5 is due today**
- **Homework 6 is out**
  Due on Friday, March 3, 2006 or earlier (TA office)
- **No class on March 3, 2006**

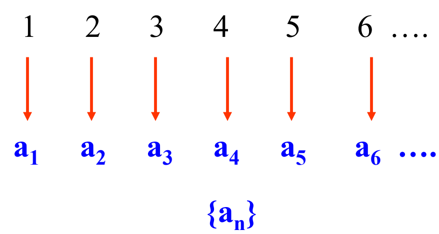
Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Sequences

Definition: A **sequence** is a function from a subset of the set of integers (typically the set {0,1,2,...} or the set {1,2,3,...} to a set S. We use the notation an to denote the image of the integer n. We call an a term of the sequence.

Notation: $\{a_n\}$ is used to represent the sequence (note {} is the same notation used for sets, so be careful). $\{a_n\}$ represents the ordered list $a_1, a_2, a_3, \ldots$.



Arithmetic progression

Definition: An **arithmetic progression** is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd$$

where a is the *initial term* and d is *common difference*, such that both belong to R.

Example:

- $s_n = -1+4n$
- members: -1, 3, 7, 11, …

Geometric progression

Definition A **geometric progression** is a sequence of the form:

$a, ar, ar^2, ..., ar^k,$

where a is the *initial term*, and r is the *common ratio*. Both a and r belong to R.

Example:

- $a_n = (½)^n$
  members: 1,½, ¼, 1/8, …..

Summations

Summation of the terms of a sequence:

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + ... + a_n$$

The variable j is referred to as the index of summation.

- m is the lower limit and
- n is the upper limit of the summation.

Summations

Example:

- **1) Sum the first 7 terms of {n^2} where n=1,2,3, ...**

$$\sum_{j=1}^{7} a_j = \sum_{j=1}^{7} j^2 =$$

Summations

Example:

- **1) Sum the first 7 terms of {n^2} where n=1,2,3, ...**

$$\sum_{j=1}^{7} a_j = \sum_{j=1}^{7} j^2 = 1 + 4 + 16 + 25 + 36 + 49 = 140$$

Summations

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- $$\sum_{j=1}^{7} a_j = \sum_{j=1}^{7} j^2 = 1 + 4 + 16 + 25 + 36 + 49 = 140$$

- **2) What is the value of**

$$\sum_{k=4}^{8} a_j = \sum_{k=4}^{8} (-1)^j =$$

Summations

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- **2) What is the value of**

$$\sum_{k=4}^{8} a_j = \sum_{k=4}^{8} (-1)^j = 1 + (-1) + 1 + (-1) + 1 = 1$$

Arithmetic series

Definition: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is called an **arithmetic series**.

Theorem: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is

$$S = \sum_{j=1}^{n}(a + jd) = na + d\sum_{j=1}^{n} j = na + d\frac{n(n+1)}{2}$$

- **Why?**

Arithmetic series

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Proof:

$$S = \sum_{j=1}^{n}(a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d\sum_{j=1}^{n} j$$

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$$\sum_{j=1}^{n} j = 1 + 2 + 3 + 4 + ... + (n-2) + (n-1) + n$$

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1+(n-1)=n

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$$S = \sum_{j=1}^{n} (a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d\sum_{j=1}^{n} j$$

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + 4 + \ldots. + (n-2) + (n-1) + n$$

1+(n-1)=n n … n

$$\frac{(n+1)}{2} * n$$

Arithmetic series

Example: $S = \sum_{j=1}^{5} (2 + j3) =$

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$$S = \sum_{j=1}^{5} (2 + j3) =$$

$$= \sum_{j=1}^{5} 2 + \sum_{j=1}^{5} j3 =$$

$$= 2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j =$$

$$= 2 * 5 + 3\sum_{j=1}^{5} j =$$

Arithmetic series

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$$= 2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j =$$

$$= 2 * 5 + 3\sum_{j=1}^{5} j =$$

$$= 10 + 3\frac{(5+1)}{2} * 5 =$$

Arithmetic series

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$$S = \sum_{j=1}^{5} (2 + j3) =$$

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$$= 2 * 5 + 3\sum_{j=1}^{5} j =$$

$$= 10 + 3\frac{(5+1)}{2} * 5 =$$

$$= 10 + 45 = 55$$

Arithmetic series

Example 2: $S = \sum_{j=3}^{5} (2 + j3) =$

Arithmetic series

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$$= \left[\sum_{j=1}^{5} (2 + j3) \right] - \left[\sum_{j=1}^{2} (2 + j3) \right]$$

⇐ **Trick**

Arithmetic series

Example 2: $S = \sum_{j=3}^{5} (2 + j3) =$

$$= \left[\sum_{j=1}^{5} (2 + j3) \right] - \left[\sum_{j=1}^{2} (2 + j3) \right]$$

Trick

$$= \left[2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j \right] - \left[2\sum_{j=1}^{2} 1 + 3\sum_{j=1}^{2} j \right]$$

$$= 55 - 13 = 42$$

Double summations

Example: $S = \sum_{i=1}^{4} \sum_{j=1}^{2} (2i - j) =$

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Double summations

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$$= \sum_{i=1}^{4} \left[\sum_{j=1}^{2} 2i - \sum_{j=1}^{2} j \right] =$$

$$= \sum_{i=1}^{4} \left[2i \sum_{j=1}^{2} 1 - \sum_{j=1}^{2} j \right] =$$

Double summations

Example: $S=\sum_{i=1}^{4}\sum_{j=1}^{2}(2i-j)=$

$$=\sum_{i=1}^{4}\left[\sum_{j=1}^{2}2i-\sum_{j=1}^{2}j\right]=$$

$$=\sum_{i=1}^{4}\left[2i\sum_{j=1}^{2}1-\sum_{j=1}^{2}j\right]=$$

$$=\sum_{i=1}^{4}\left[2i*2-\sum_{j=1}^{2}j\right]=$$

$$=\sum_{i=1}^{4}\left[2i*2-3\right]=$$

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$$=\sum_{i=1}^{4}\left[2i*2-3\right]=$$

$$=\sum_{i=1}^{4}4i-\sum_{i=1}^{4}3=$$

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$$= \sum_{i=1}^{4}\left[2i * 2 - \sum_{j=1}^{2} j\right] =$$

$$= \sum_{i=1}^{4}[2i * 2 - 3] =$$

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$$= 4\sum_{i=1}^{4} i - 3\sum_{i=1}^{4} 1 =$$

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$$= 4\sum_{i=1}^{4} i - 3\sum_{i=1}^{4} 1 = \quad 4 * 10 - 3 * 4 = 28$$

Geometric series

Definition: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^k is called **a geometric series**.

Theorem: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^n is

$$S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left[\frac{r^{n+1} - 1}{r - 1} \right]$$

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Proof:

$$S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + \ldots + ar^n$$

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- multiply S by r

$$rS = r\sum_{j=0}^{n} ar^j = ar + ar^2 + ar^3 + \ldots + ar^{n+1}$$

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$$S = \frac{ar^{n+1} - a}{r-1} = a\left[\frac{r^{n+1}-1}{r-1}\right]$$

Geometric series

Example:

$$S = \sum_{j=0}^{n} 2(5)^j =$$

Geometric series

Example:

$$S = \sum_{j=0}^{3} 2(5)^j =$$

General formula:

$$S = \sum_{j=0}^{n} (ar^j) = a\sum_{j=0}^{n} r^j = a\left[\frac{r^{n+1}-1}{r-1}\right]$$

$$S = \sum_{j=0}^{3} 2(5)^j = 2*\frac{5^4-1}{5-1} =$$

$$= 2*\frac{625-1}{4} = 2*\frac{624}{4} = 2*156 = 312$$

Summations

Summation:

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n$$

The variable j is referred to as the index of summation.

- m is the lower limit and
- n is the upper limit of the summation.

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Arithmetic series

How to calculate:

$$S = \sum_{j=4}^{7} (3 + 5j) =$$

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$$S = \frac{ar^{n+1}-a}{r-1} = a\left[\frac{r^{n+1}-1}{r-1}\right]$$

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$$S = \sum_{j=0}^{3} 2(5)^j = 2 * \frac{5^4 - 1}{5 - 1} =$$

$$= 2 * \frac{625-1}{4} = 2 * \frac{624}{4} = 2 * 156 = 312$$

Infinite geometric series

- Infinite geometric series can be computed in the closed form for x<1

$$S = \sum_{n=0}^{\infty} (x^n)$$

- How?

Infinite geometric series

- Infinite geometric series can be computed in the closed form for x<1
- How?

$$\sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} \sum_{n=0}^{k} x^n = \lim_{k \to \infty} \frac{x^{k+1} - 1}{x - 1} = -\frac{1}{x-1} = \frac{1}{1-x}$$

- Thus:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$