Summations

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Course administration

• Homework 5 is due today

• Homework 6 is out
  Due on Friday, March 3, 2006 or earlier (TA office)

• No class on March 3, 2006

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/
Sequences

**Definition:** A sequence is a function from a subset of the set of integers (typically the set \( \{0, 1, 2, \ldots\} \) or the set \( \{1, 2, 3, \ldots\} \) to a set \( S \). We use the notation \( a_n \) to denote the image of the integer \( n \). We call \( a_n \) a term of the sequence.

**Notation:** \( \{a_n\} \) is used to represent the sequence (note \{\} is the same notation used for sets, so be careful). \( \{a_n\} \) represents the ordered list \( a_1, a_2, a_3, \ldots \).

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \ldots \\
\{a_n\}
\end{array}
\]

Arithmetic progression

**Definition:** An arithmetic progression is a sequence of the form \( a, a+d, a+2d, \ldots, a+nd \) where \( a \) is the initial term and \( d \) is common difference, such that both belong to \( \mathbb{R} \).

**Example:**
- \( s_n = -1 + 4n \)
- members: \(-1, 3, 7, 11, \ldots \)
Geometric progression

**Definition** A geometric progression is a sequence of the form:

\[ a, ar, ar^2, \ldots, ar^k, \]

where \( a \) is the initial term, and \( r \) is the common ratio. Both \( a \) and \( r \) belong to \( \mathbb{R} \).

**Example:**

- \( a_n = \left( \frac{1}{2} \right)^n \)
  - members: 1, \( \frac{1}{2} \), \( \frac{1}{4} \), \( \frac{1}{8} \), …

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Summations

**Summation of the terms of a sequence:**

\[
\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n
\]

The variable \( j \) is referred to as the index of summation.

- \( m \) is the lower limit and
- \( n \) is the upper limit of the summation.
Summations

Example:

• 1) Sum the first 7 terms of \( \{n^2\} \) where \( n=1,2,3, \ldots \).

\[
\sum_{j=1}^{7} a_j = \sum_{j=1}^{7} j^2 =
\]

\[
= 1 + 4 + 16 + 25 + 36 + 49 = 140
\]
Summations

Example:

• 1) Sum the first 7 terms of \( \{n^2\} \) where \( n=1,2,3, \ldots \)

\[
\sum_{j=1}^{7} a_j = \sum_{j=1}^{7} j^2 = 1 + 4 + 16 + 25 + 36 + 49 = 140
\]

• 2) What is the value of

\[
\sum_{k=4}^{8} a_j = \sum_{k=4}^{8} (-1)^j =
\]
Arithmetic series

**Definition:** The sum of the terms of the arithmetic progression \( a, a+d, a+2d, \ldots, a+nd \) is called an arithmetic series.

**Theorem:** The sum of the terms of the arithmetic progression \( a, a+d, a+2d, \ldots, a+nd \) is

\[
S = \sum_{j=1}^{n} (a + jd) = na + d \sum_{j=1}^{n} j = na + d \frac{n(n+1)}{2}
\]

- Why?

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Arithmetic series

**Theorem:** The sum of the terms of the arithmetic progression \( a, a+d, a+2d, \ldots, a+nd \) is

\[
S = \sum_{j=1}^{n} (a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d \sum_{j=1}^{n} j
\]

**Proof:**

\[
S = \sum_{j=1}^{n} (a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d \sum_{j=1}^{n} j
\]
Arithmetic series

**Theorem:** The sum of the terms of the arithmetic progression 
\( a, a+d, a+2d, \ldots, a+nd \) is 
\[
S = \sum_{j=1}^{n} (a + jd) = na + d \sum_{j=1}^{n} j = na + d \frac{n(n+1)}{2}
\]

**Proof:**
\[
S = \sum_{j=1}^{n} (a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d \sum_{j=1}^{n} j
\]
\[
\sum_{j=1}^{n} j = 1 + 2 + 3 + 4 + \ldots + (n-2) + (n-1) + n
\]
\[
1 + (n-1) = n
\]
**Arithmetic series**

**Theorem:** The sum of the terms of the arithmetic progression $a, a+d, a+2d, \ldots, a+nd$ is

$$S = \sum_{j=1}^{n} (a + jd) = na + d \sum_{j=1}^{n} j = na + d \frac{n(n+1)}{2}$$

**Proof:**

$$S = \sum_{j=1}^{n} (a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d \sum_{j=1}^{n} j$$

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + 4 + \ldots + (n-2) + (n-1) + n$$

$$1 + (n-1) = n, \quad n, \quad \ldots, \quad n$$
Arithmetic series

Example:

\[ S = \sum_{j=1}^{5} (2 + j3) = \]
Arithmetic series

Example: \( S = \sum_{j=1}^{5} (2 + j3) = \)
\( = \sum_{j=1}^{5} 2 + \sum_{j=1}^{5} j3 = \)
\( = 2 \sum_{j=1}^{5} 1 + 3 \sum_{j=1}^{5} j = \)
\( = 2 \times 5 + 3 \times 10 = \)

\( = 2 \times 5 + 3 \times 10 = \)
Arithmetic series

Example:

\[
S = \sum_{j=1}^{5} (2 + j3) = \\
= \sum_{j=1}^{5} 2 + \sum_{j=1}^{5} j3 = \\
= 2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j = \\
= 2 \cdot 5 + 3 \sum_{j=1}^{5} j = \\
= 10 + 3 \left(\frac{5+1}{2}\right) \cdot 5 = \\
= 10 + 45 = 55
\]
Arithmetic series

Example 2:

\[ S = \sum_{j=3}^{5} (2 + j3) = \]

\[ = \left[ \sum_{j=1}^{5} (2 + j3) \right] - \left[ \sum_{j=1}^{2} (2 + j3) \right] \quad \text{Trick} \]
### Arithmetic series

**Example 2:**

\[
S = \sum_{j=3}^{5} (2 + j3) = \\
= \left[ \sum_{j=1}^{5} (2 + j3) \right] - \left[ \sum_{j=1}^{2} (2 + j3) \right] \quad \text{Trick}
\]

\[
= \left[ 2 \sum_{j=1}^{5} 1 + 3 \sum_{j=1}^{5} j \right] - \left[ 2 \sum_{j=1}^{2} 1 + 3 \sum_{j=1}^{2} j \right]
\]

\[
= 55 - 13 = 42
\]

### Double summations

**Example:**

\[
S = \sum_{i=1}^{4} \sum_{j=1}^{2} (2i - j) = 
\]
Double summations

Example: \[ S = \sum_{i=1}^{4} \sum_{j=1}^{2} (2i - j) = \]
\[ = \sum_{j=1}^{4} \left( \sum_{j=1}^{2} 2i - \sum_{j=1}^{2} j \right) = \]
\[ = \sum_{j=1}^{4} 2i \sum_{j=1}^{2} 1 - \sum_{j=1}^{2} j = \]
Double summations

Example: $S = \sum_{i=1}^{4} \sum_{j=1}^{2} (2i - j) =$

$= \sum_{i=1}^{4} \left[ \sum_{j=1}^{2} 2i - \sum_{j=1}^{2} j \right] =$

$= \sum_{i=1}^{4} \left[ 2i \sum_{j=1}^{2} 1 - \sum_{j=1}^{2} j \right] =$

$= \sum_{i=1}^{4} \left[ 2i \cdot 2 - \sum_{j=1}^{2} j \right] =$

$= \sum_{i=1}^{4} [2i \cdot 2 - 3] =$

$= \sum_{i=1}^{4} [4i - 3] =$
Double summations

Example:  \[ S = \sum_{i=1}^{4} \sum_{j=1}^{2} (2i - j) = \]
\[ = \sum_{i=1}^{4} \left[ \sum_{j=1}^{2} 2i - \sum_{j=1}^{2} j \right] = \]
\[ = \sum_{i=1}^{4} \left[ 2i \sum_{j=1}^{2} 1 - \sum_{j=1}^{2} j \right] = \]
\[ = \sum_{i=1}^{4} \left[ 2i \cdot 2 - \sum_{j=1}^{2} j \right] = \]
\[ = \sum_{i=1}^{4} [2i \cdot 2 - 3] = \]
\[ = \sum_{i=1}^{4} 4i - \sum_{i=1}^{4} 3 = \]
\[ = 4 \sum_{i=1}^{4} i - 3 \sum_{i=1}^{4} 1 = \]
\[ = 4 \cdot 10 - 3 \cdot 4 = 28 \]
Geometric series

**Definition:** The sum of the terms of a geometric progression $a$, $ar$, $ar^2$, ..., $ar^k$ is called a geometric series.

**Theorem:** The sum of the terms of a geometric progression $a$, $ar$, $ar^2$, ..., $ar^n$ is

$$S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left[ \frac{r^{n+1} - 1}{r-1} \right]$$

**Proof:**

$$S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + ... + ar^n$$
Geometric series

Theorem: The sum of the terms of a geometric progression $a$, $ar$, $ar^2$, ..., $ar^n$ is

$$S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left( \frac{r^{n+1} - 1}{r - 1} \right)$$

Proof:

$$S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + ... + ar^n$$

- multiply $S$ by $r$

$$rS = r \sum_{j=0}^{n} ar^j = ar + ar^2 + ar^3 + ... + ar^{n+1}$$

- Substract $rS - S = [ar + ar^2 + ar^3 + ... + ar^{n+1}] - [a + ar + ar^2 + ... + ar^n]$
Geometric series

**Theorem:** The sum of the terms of a geometric progression \(a, ar, ar^2, \ldots, ar^n\) is

\[ S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \frac{r^{n+1} - 1}{r - 1} \]

**Proof:**

\[ S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + \ldots + ar^n \]

- multiply \(S\) by \(r\)

\[ rS = r \sum_{j=0}^{n} ar^j = ar + ar^2 + ar^3 + \ldots + ar^{n+1} \]

- Subtract \(rS - S = \left[ ar + ar^2 + ar^3 + \ldots + ar^{n+1} \right] - \left[ a + ar + ar^2 + \ldots + ar^n \right] = ar^{n+1} - a \)

\[ S = \frac{ar^{n+1} - a}{r - 1} = a \frac{r^{n+1} - 1}{r - 1} \]
Geometric series

Example:

\[ S = \sum_{j=0}^{n} 2(5)^j = \]

General formula:

\[ S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left( \frac{r^{n+1} - 1}{r - 1} \right) \]

\[ S = \sum_{j=0}^{3} 2(5)^j = 2 \cdot \frac{5^4 - 1}{5 - 1} = \]

\[ = 2 \cdot \frac{625 - 1}{4} = 2 \cdot 624 = 2 \cdot 156 = 312 \]
Summations

**Summation:**

\[ \sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n \]

The variable \( j \) is referred to as the index of summation.

- \( m \) is the lower limit and
- \( n \) is the upper limit of the summation.

Arithmetic series

**Definition:** The sum of the terms of the arithmetic progression \( a, a+d, a+2d, \ldots, a+nd \) is called an arithmetic series.

**Theorem:** The sum of the terms of the arithmetic progression \( a, a+d, a+2d, \ldots, a+nd \) is

\[ S = \sum_{j=1}^{n} (a + jd) = na + d \sum_{j=1}^{n} j = na + d \frac{n(n+1)}{2} \]
Arithmetic series

How to calculate:

\[ S = \sum_{j=1}^{7} (3 + 5j) = \]

Geometric series

**Definition:** The sum of the terms of a geometric progression \( a, ar, ar^2, ..., ar^k \) is called a geometric series.

**Theorem:** The sum of the terms of a geometric progression \( a, ar, ar^2, ..., ar^n \) is

\[
S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left[ \frac{1}{r} - \frac{r^{n+1} - 1}{r - 1} \right]
\]
Geometric series

**Theorem:** The sum of the terms of a geometric progression \(a, ar, ar^2, ..., ar^n\) is

\[
S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left[ \frac{r^{n+1} - 1}{r - 1} \right]
\]

**Proof:**

\[
S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + ... + ar^n
\]

- multiply \(S\) by \(r\)

\[
rS = r \sum_{j=0}^{n} ar^j = ar + ar^2 + ar^3 + ... + ar^{n+1}
\]
**Theorem:** The sum of the terms of a geometric progression \( a, ar, ar^2, \ldots, ar^n \) is

\[
S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \frac{r^{n+1} - 1}{r - 1}
\]

**Proof:**

\[
S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + \ldots + ar^n
\]

- multiply \( S \) by \( r \)
  \[rS = r \sum_{j=0}^{n} ar^j = ar + ar^2 + ar^3 + \ldots + ar^{n+1}\]
- Subtract \( rS - S = [ar + ar^2 + ar^3 + \ldots + ar^{n+1}] - [a + ar + ar^2 + \ldots + ar^n] \)

\[= ar^{n+1} - a\]
Geometric series

**Theorem:** The sum of the terms of a geometric progression $a, ar, ar^2, \ldots, ar^n$ is

$$S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left[ \frac{r^{n+1} - 1}{r - 1} \right]$$

**Proof:**

- $S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + \ldots + ar^n$
- multiply $S$ by $r$
  $$rS = r \sum_{j=0}^{n} ar^j = ar + ar^2 + ar^3 + \ldots + ar^{n+1}$$
- Subtract $rS - S = [ar + ar^2 + ar^3 + \ldots + ar^{n+1}] - [a + ar^2 + \ldots + ar^n]$
  $$= ar^{n+1} - a$$
  $$S = \frac{ar^{n+1} - a}{r - 1} = a \left[ \frac{r^{n+1} - 1}{r - 1} \right]$$

**Example:**

$$S = \sum_{j=0}^{n} 2(5)^j =$$
Geometric series

Example:

\[ S = \sum_{j=0}^{3} 2(5)^j = \]

General formula:

\[ S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left[ \frac{r^{n+1} - 1}{r - 1} \right] \]
Infinite geometric series

- Infinite geometric series can be computed in the closed form for $x<1$
  \[ S = \sum_{n=0}^{\infty} (x^n) \]

- How?

Thus:

\[
\sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} \sum_{n=0}^{k} x^n = \lim_{k \to \infty} \frac{x^{k+1} - 1}{x - 1} = \frac{1}{x - 1} = \frac{1}{1 - x}
\]

- Thus:

\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}
\]