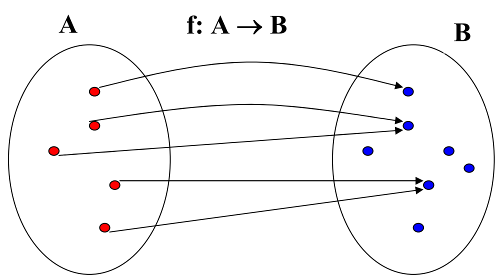
CS 441 Discrete Mathematics for CS
Lecture 12

Functions

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

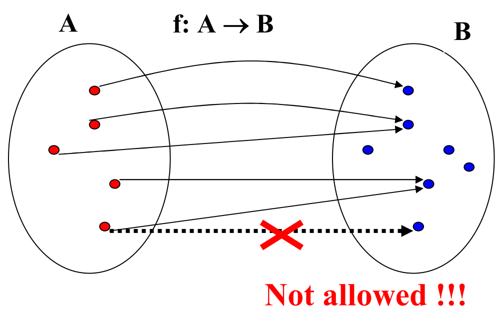
Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted **f : A → B** , is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted **$f : A \rightarrow B$**, is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ to denote the assignment of b to an element a of A by the function f.



Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example1:

- Let A = {1,2,3} and B = {a,b,c}
- Assume f is defined as:
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$
- Is f a function ?

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example1:

- Let A = {1,2,3} and B = {a,b,c}
- Assume f is defined as:
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$
- Is f a function ?
- **Yes.** since f(1)=c, f(2)=a, f(3)=c. each element of A is assigned an element from B

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 2:

- Let A = {1,2,3} and B = {a,b,c}
- Assume g is defined as
  - $1 \rightarrow c$
  - $1 \rightarrow b$
  - $2 \rightarrow a$
  - $3 \rightarrow c$
- Is g a function?

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 2:

- Let A = {1,2,3} and B = {a,b,c}
- Assume g is defined as
  - $1 \to c$
  - $1 \to b$
  - $2 \to a$
  - $3 \to c$
- Is g a function?
- **No.** g(1) = is assigned both c and b.

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 3:

- A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}
- Define h: $A \to B$ as:
  - h(x) = x mod 3.
  - (the result is the remainder after the division by 3)
- Assignments:
- $0 \to$?

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 3:

- A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}
- Define h: A → B as:
  - h(x) = x mod 3.
  - (the result is the remainder after the division by 3)
- Assignments:
- 0 → 0
- 1 → ?

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 3:

- A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}
- Define h: A → B as:
  - h(x) = x mod 3.
  - (the result is the remainder after the division by 3)
- Assignments:
- 0 → 0
- 1 → 1
- 2 → ?

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 3:

- A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}
- Define h: A → B as:
  - h(x) = x mod 3.
  - (the result is the remainder after the division by 3)
- Assignments:
- 0 → 0 3 → ?
- 1 → 1
- 2 → 2

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 3:

- A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}
- Define h: A → B as:
  - h(x) = x mod 3.
  - (the result is the remainder after the division by 3)
- Assignments:
- 0 → 0 3 → 0
- 1 → 1 4 → ?
- 2 → 2

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 3:

- $A = \{0,1,2,3,4,5,6,7,8,9\}$, $B = \{0,1,2\}$
- Define $h: A \to B$ as:
  - $h(x) = x \bmod 3$.
  - (the result is the remainder after the division by 3)
- Assignments:
- $0 \to 0$ $\quad$ $3 \to 0$
- $1 \to 1$ $\quad$ $4 \to 1$
- $2 \to 2$ $\quad$ …

Important sets in discrete math

Definitions: Let f be a function from A to B. We say that A is the **domain** of f and B is the **codomain** of f. If f(a) = b, we say that **b is the image of a** and **a is a pre-image of b**. **The range of f** is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let **A = {1,2,3} and B = {a,b,c}**

- Assume f is defined as: **1 → c, 2 → a, 3 → c**
- What is the image of 1?

Important sets in discrete math

Definitions: Let f be a function from A to B. We say that A is the **domain** of f and B is the **codomain** of f. If f(a) = b, we say that **b is the image of a** and **a is a pre-image of b**. **The range of f** is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let A = **{1,2,3} and B = {a,b,c}**

- Assume f is defined as: **1 → c, 2 → a, 3 → c**
- What is the image of 1?
- 1 → c c is the image of 1
- What is the pre-image of a?

Important sets in discrete math

Definitions: Let f be a function from A to B. We say that A is the **domain** of f and B is the **codomain** of f. If f(a) = b, we say that **b is the image of a** and **a is a pre-image of b**. **The range of f** is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let A = **{1,2,3} and B = {a,b,c}**

- Assume f is defined as: **1 → c, 2 → a, 3 → c**
- What is the image of 1?
- 1 → c c is the image of 1
- What is the pre-image of a?
- 2 → a 2 is a pre-image of a.
- Domain of f ?

Important sets in discrete math

Definitions: Let f be a function from A to B. We say that A is the **domain** of f and B is the **codomain** of f. If f(a) = b, we say that **b is the image of a** and **a is a pre-image of b**. **The range of f** is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let **A = {1,2,3} and B = {a,b,c}**

- Assume f is defined as: **1 → c, 2 → a, 3 → c**
- What is the image of 1?
- 1 → c c is the image of 1
- What is the pre-image of a?
- 2 → a 2 is a pre-image of a.
- Domain of f ? {1,2,3}
- Codomain of f ?

Important sets in discrete math

Definitions: Let f be a function from A to B. We say that A is the **domain** of f and B is the **codomain** of f. If f(a) = b, we say that **b is the image of a** and **a is a pre-image of b**. **The range of f** is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let **A = {1,2,3} and B = {a,b,c}**

- Assume f is defined as: **1 → c, 2 → a, 3 → c**
- What is the image of 1?
- 1 → c c is the image of 1
- What is the pre-image of a?
- 2 → a 2 is a pre-image of a.
- Domain of f ? {1,2,3}
- Codomain of f ? {a,b,c}
- Range of f ?

Important sets in discrete math

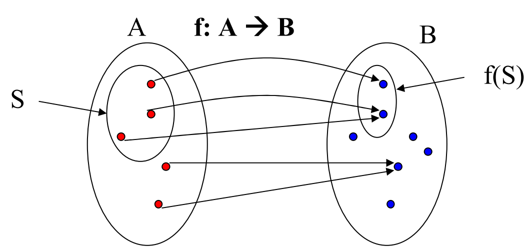
Definitions: Let f be a function from A to B. We say that A is the **domain** of f and B is the **codomain** of f. If f(a) = b, we say that **b is the image of a** and **a is a pre-image of b**. **The range of f** is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let A = {1,2,3} and B = {a,b,c}

- Assume f is defined as: 1 → c, 2 → a, 3 → c
- What is the image of 1?
- 1 → c   c is the image of 1
- What is the pre-image of a?
- 2 → a   2 is a pre-image of a.
- Domain of f ? {1,2,3}
- Codomain of f ? {a,b,c}
- Range of f ? {a,c}

Image of a subset

Definition: Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that f(S) = { f(s) | s ∈ S }.

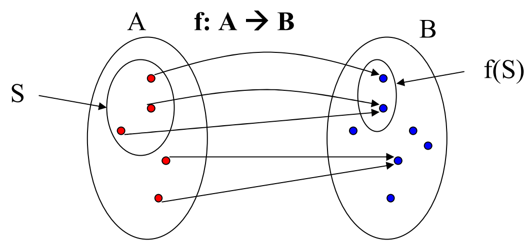


Example:

- Let A = {1,2,3} and B = {a,b,c} and f: 1 → c, 2 → a, 3 → c
- Let S = {1,3} then image f(S) = **?**

Image of a subset

Definition: Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that $f(S) = \{ f(s) \mid s \in S \}$.



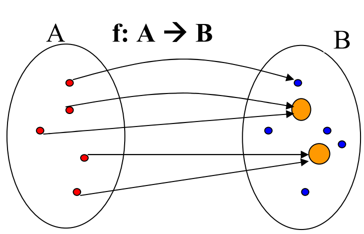
Example:

- Let A = {1,2,3} and B = {a,b,c} and f: 1 → c, 2 → a, 3 → c
- Let S = {1,3} then image f(S) = {c}.

Injective function

Definition: A function f is said to be **one-to-one, or injective**, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an **injection if it is one-to-one**.

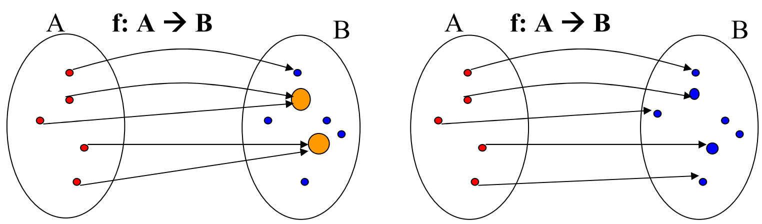
Alternative: A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y. This is the contrapositive of the definition.



Injective function

Definition: A function f is said to be **one-to-one, or injective**, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an **injection if it is one-to-one**.

Alternate: A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y. This is the contrapositive of the definition.



Not injective **Injective function**

Injective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - 1 → c
  - 2 → a
  - 3 → c
- Is f one to one?

Injective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$
- Is f one to one? **No**, it is not one-to-one since f(1) = f(3) = c, and $1 \neq 3$.

Example 2: Let $g : Z \rightarrow Z$, where g(x) = 2x - 1.

- Is g is one-to-one (why?)

Injective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$
- Is f one to one? **No**, it is not one-to-one since f(1) = f(3) = c, and $1 \neq 3$.

Example 2: Let $g : Z \rightarrow Z$, where g(x) = 2x - 1.

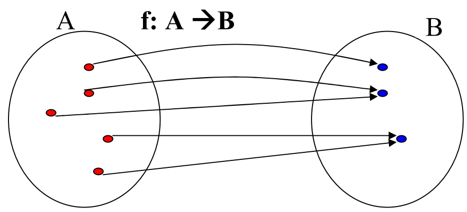
- Is g is one-to-one (why?)
- **Yes.**
- Suppose g(a) = g(b), i.e., 2a - 1 = 2b - 1 => 2a = 2b

`` => a = b.

Surjective function

Definition: A function f from A to B is called **onto**, or **surjective**, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

Alternative: all co-domain elements are covered



Surjective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

– Define f as
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$

- Is f onto?

Surjective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - $1 \to c$
  - $2 \to a$
  - $3 \to c$
- Is f onto?
- **No.** f is not onto, since $b \in B$ has no pre-image.

Example 2: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

- Define h: $A \to B$ as h(x) = x mod 3.
- Is h onto function?

Surjective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

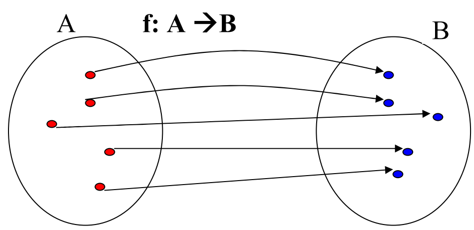
- Define f as
  - $1 \to c$
  - $2 \to a$
  - $3 \to c$
- Is f an onto?
- **No.** f is not onto, since $b \in B$ has no pre-image.

Example 2: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

- Define h: $A \to B$ as h(x) = x mod 3.
- Is h an onto function?
- **Yes.** h is onto since a pre-image of 0 is 6, a pre-image of 1 is 4, a pre-image of 2 is 8.

Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one and onto**.



Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - $1 \rightarrow c$
    - $2 \rightarrow a$
    - $3 \rightarrow b$
- Is f is a bijection? **Yes.** It is both one-to-one and onto.

- **Note:** Let f be a function from a set A to itself, where A is finite. f is one-to-one if and only if f is onto.
- This is not true for A an infinite set. Define $f : Z \rightarrow Z$, where $f(z) = 2 * z$. f is one-to-one but not onto (3 has no pre-image).

Bijective functions

Example 2:

- Define g : W → W (whole numbers), where
  g(n) = [n/2] (floor function).
    - 0 → [0/2] = [0] = 0
    - 1 → [1/2] = [1/2] = 0
    - 2 → [2/2] = [1] = 1
    - 3 → [3/2] = [3/2] = 1
- ...
- Is g a bijection?

Bijective functions

Example 2:

- Define g : W → W (whole numbers), where
  g(n) = [n/2] (floor function).
    - 0 → [0/2] = [0] = 0
    - 1 → [1/2] = [1/2] = 0
    - 2 → [2/2] = [1] = 1
    - 3 → [3/2] = [3/2] = 1
- ...
- Is g a bijection?
    - **No.** g is onto but not 1-1 (g(0) = g(1) = 0 however 0 ≠ 1.