CS 441 Discrete Mathematics for CS
Lecture 11

Sets and set operations

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Course administration

Homework 3:
- **Due today**

Homework 4:
- **Due next week on Friday, February 10, 2006**

Midterm 1:
- **Wednesday, February 15, 2006**
- **Covers chapter 1 of the textbook**
- **Closed book**
- **Tables for equivalences and rules of inference will be given to you**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Review

Definition: A **set** is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set. (Cantor's naive definition)

Example: First seven prime numbers.

X = { 2, 3, 5, 7, 11, 13, 17 }

Definition: An **ordered n-tuple** (x1, x2, ..., xN) is the ordered collection that has x1 as its first element, x2 as its second element, ..., and xN as its N-th element, N ≥ 2.

Example:

- Coordinates of a point in the 2-D plane (12, 16)

Cartesian product

Definition: Let S and T be sets. The **Cartesian product of S and T**, denoted by **S x T,** is the set of all ordered pairs (s,t), where s ∈ S and t ∈ T. Hence,

- $S \times T = \{ (s,t) \mid s \in S \wedge t \in T \}$.

Examples:

- S = {1,2} and T = {a,b,c}
- S x T = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }
- T x S = { (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) }
- Note: S x T ≠ T x S **!!!!**

Cardinality of the Cartesian product

- |S x T| = |S| * |T|.

Example:

- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}
- A x B=

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- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}
- A x B= {(John, Jane),(John, Ann) , (John, Laura), (Peter, Jane), (Peter, Ann) , (Peter, Laura) , (Mike, Jane) , (Mike, Ann) , (Mike, Laura)}
- |A x B| =

Cardinality of the Cartesian product

- |S x T| = |S| * |T|.

Example:

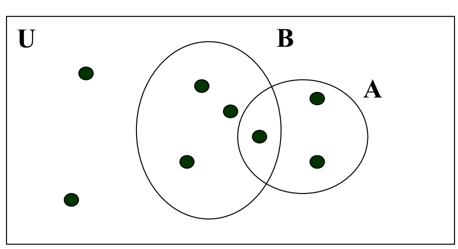
- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}
- A x B= {(John, Jane), (John, Ann) , (John, Laura), (Peter, Jane), (Peter, Ann) , (Peter, Laura) , (Mike, Jane) , (Mike, Ann) , (Mike, Laura)}
- |A x B| = 9
- |A|=3, |B|=3 → |A| |B|= 9

Definition: A subset of the Cartesian product A x B is called a relation from the set A to the set B.

Set operations

Definition: Let A and B be sets. The **union of A and B**, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

- Alternate: $A \cup B = \{ x \mid x \in A \vee x \in B \}$.



- **Example:**
- A = {1,2,3,6} B = { 2,4,6,9}
- $A \cup B$ = ?

Set operations

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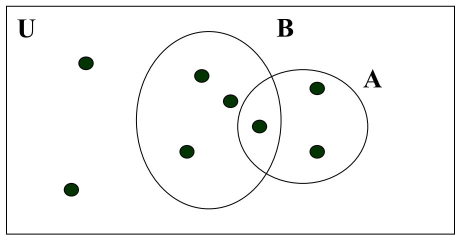
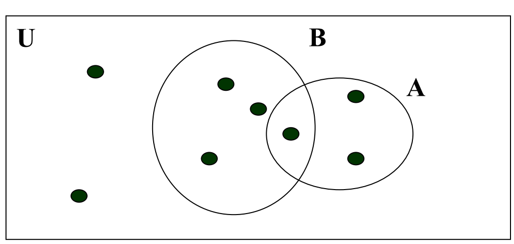


- **Example:**
- $A = \{1,2,3,6\}$ $B = \{ 2,4,6,9\}$
- $A \cup B = \{ 1,2,3,4,6,9 \}$

Set operations

Definition: Let A and B be sets. The **intersection of A and B**, denoted by $A \cap B$, is the set that contains those elements that are in both A and B.

- Alternate: $A \cap B = \{ x \mid x \in A \land x \in B \}$.



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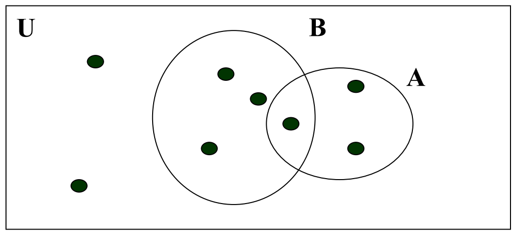


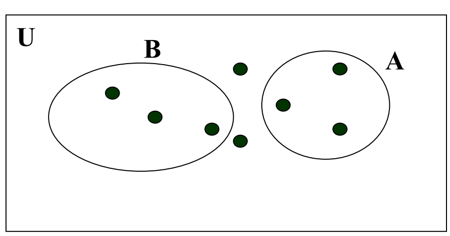
Example:

- A = {1,2,3,6}   B = { 2, 4, 6, 9}
- $A \cap B = \{ 2, 6 \}$

Disjoint sets

Definition: Two sets are called **disjoint** if their intersection is empty.

- Alternate: A and B are disjoint **if and only if** $A \cap B = \emptyset$.



Example:

- A={1,2,3,6}   B={4,7,8}   Are these disjoint?

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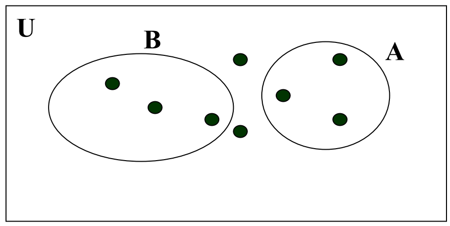


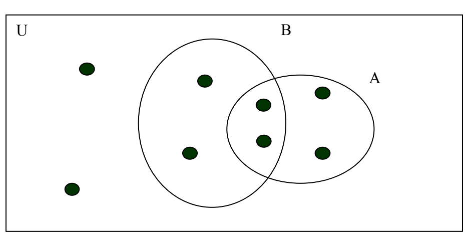
Example:

- A={1,2,3,6} B={4,7,8} Are these disjoint?
- Yes.
- $A \cap B = \emptyset$

Cardinality of the set union

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$

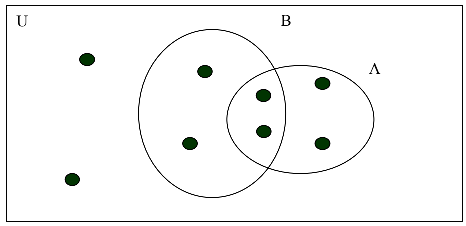


- Why this formula?

Cardinality of the set union

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$

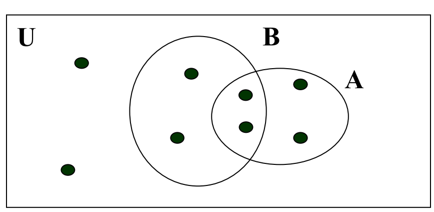


- Why this formula? Correct for an over-count.
- More general rule:
  - **The principle of inclusion and exclusion.**

Set difference

Definition: Let A and B be sets. The **difference of A and B**, denoted by **A - B**, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

- Alternate: $A - B = \{ x \mid x \in A \wedge x \notin B \}$.



Example: A= {1,2,3,5,7} B = {1,5,6,8}

- A - B =?

Set difference

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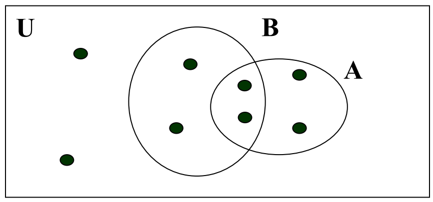


Example: A= {1,2,3,5,7} B = {1,5,6,8}

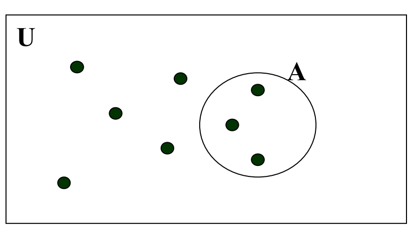
- A - B ={2,3,7}

Complement of a set

Definition: Let U be the **universal set**: the set of all objects under the consideration.

Definition: The **complement of the set A**, denoted by $\bar{A}$, is the complement of A with respect to U.

- Alternate: $\bar{A} = \{ x \mid x \notin A \}$



Example: U={1,2,3,4,5,6,7,8} A ={1,3,5,7}

- $\bar{A}$ = ?

Complement of a set

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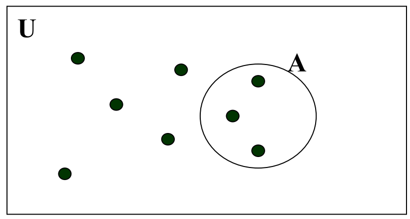

Example: U={1,2,3,4,5,6,7,8} A ={1,3,5,7}

- $\overline{A}$ = {2,4,6,8}

Set identities

Set Identities (analogous to logical equivalences)

- **Identity**
  - $A \cup \varnothing = A$
  - $A \cap U = A$
- **Domination**
  - $A \cup U = U$
  - $A \cap \varnothing = \varnothing$
- **Idempotent**
  - $A \cup A = A$
  - $A \cap A = A$

Set identities

- **Double complement**
  - $\overline{\overline{A}} = A$
- **Commutative**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Set identities

- **DeMorgan**
  - $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$
  - $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
- **Absorbtion Laws**
  - $A \cup (A \cap B) = A$
  - $A \cap (A \cup B) = A$
- **Complement Laws**
  - $A \cup \overline{A} = U$
  - $A \cap \overline{A} = \emptyset$

Set identities

- **Set identities can be proved using membership tables.**
- List each combination of sets that an element can belong to. Then show that for each such a combination the element either belongs or does not belong to both sets in the identity.
- Prove: $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

A	B	$\overline{A}$	$\overline{B}$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	0	0
1	0	0	1	1	1
0	1	1	0	1	1
0	0	1	1	1	1

Generalized unions and itersections

Definition: The **union of a collection of sets** is the set that contains those elements that are members of at least one set in the collection.

$$\bigcup_{i=1}^{n} A_i = \{A_1 \cup A_2 \cup \ldots \cup A_n\}$$

Example:

- Let $A_i = \{1,2,\ldots,i\}$ i =1,2,…,n
- $$\bigcup_{i=1}^{n} A_i = \{1,2,\ldots, n\}$$

Generalized unions and intersections

Definition: The **intersection of a collection of sets** is the set that contains those elements that are members of all sets in the collection.

$$\bigcap_{i=1}^{n} A_i = \{A_1 \cap A_2 \cap \ldots \cap A_n\}$$

Example:

- Let $A_i = \{1,2,\ldots,i\}$ i =1,2,…,n

$$\bigcap_{i=1}^{n} A_i = \{1\}$$

Computer representation of sets

Idea: Assign a bit in a bit string to each element in the univesral set and set the bit to 1 if the element is present otherwise use 0

Example:

All possible elements: U={1 2 3 4 5}

- Assume A={2,5}
  - Computer representation: A = 01001
- Assume B={1,5}
  - Computer representation: B = 10001

Computer representation of sets

Example:

- A = 01001
- B = 10001

- The **union** is modeled with a bitwise **or**
- $A \vee B = 11001$
- The **intersection** is modeled with a bitwise **and**
- $A \wedge B = 00001$
- The **complement** is modeled …?

Computer representation of sets

Example:

- A = 01001
- B = 10001

- The **union** is modeled with a bitwise **or**
- $A \vee B = 11001$
- The **intersection** is modeled with a bitwise **and**
- $A \wedge B = 00001$
- The **complement** is modeled with a bitwise **negation**
- $\bar{A} = 10110$