Sets and set operations

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Course administration

Homework 3:
• Due today

Homework 4:
• Due next week on Friday, February 10, 2006

Midterm 1:
• Wednesday, February 15, 2006
• Covers chapter 1 of the textbook
• Closed book
• Tables for equivalences and rules of inference will be given to you

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/
**Review**

**Definition:** A set is a (unordered) collection of objects. These objects are sometimes called elements or members of the set. (Cantor's naive definition)

**Example:** First seven prime numbers.

\[ X = \{ 2, 3, 5, 7, 11, 13, 17 \} \]

**Definition:** An ordered n-tuple \((x_1, x_2, ..., x_N)\) is the ordered collection that has \(x_1\) as its first element, \(x_2\) as its second element, ..., and \(x_N\) as its N-th element, \(N \geq 2\).

**Example:**
- Coordinates of a point in the 2-D plane \((12, 16)\)

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**Cartesian product**

**Definition:** Let \(S\) and \(T\) be sets. The **Cartesian product of \(S\) and \(T\)**, denoted by \(S \times T\), is the set of all ordered pairs \((s, t)\), where \(s \in S\) and \(t \in T\). Hence,

\[ S \times T = \{ (s, t) \mid s \in S \land t \in T \}. \]

**Examples:**
- \(S = \{1, 2\}\) and \(T = \{a, b, c\}\)
- \(S \times T = \{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c) \}\)
- \(T \times S = \{ (a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2) \}\)
- Note: \(S \times T \neq T \times S \) !!!!
Cardinality of the Cartesian product

- \(|S \times T| = |S| \times |T|\).

**Example:**
- \(A = \{\text{John, Peter, Mike}\}\)
- \(B = \{\text{Jane, Ann, Laura}\}\)
- \(A \times B = \{(\text{John, Jane}), (\text{John, Ann}), (\text{John, Laura}), (\text{Peter, Jane}), (\text{Peter, Ann}), (\text{Peter, Laura}), (\text{Mike, Jane}), (\text{Mike, Ann}), (\text{Mike, Laura})\}\)
- \(|A \times B| = \)
Cardinality of the Cartesian product

- \(|S \times T| = |S| \times |T|\).

**Example:**
- \(A = \{\text{John, Peter, Mike}\}\)
- \(B = \{\text{Jane, Ann, Laura}\}\)
- \(A \times B = \{(\text{John, Jane}), (\text{John, Ann}), (\text{John, Laura}), (\text{Peter, Jane}), (\text{Peter, Ann}), (\text{Peter, Laura}), (\text{Mike, Jane}), (\text{Mike, Ann}), (\text{Mike, Laura})\}\)
- \(|A \times B| = 9\)
- \(|A| = 3, |B| = 3 \Rightarrow |A| \times |B| = 9\)

**Definition:** A subset of the Cartesian product \(A \times B\) is called a relation from the set \(A\) to the set \(B\).

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Set operations

**Definition:** Let \(A\) and \(B\) be sets. The **union of \(A\) and \(B\)**, denoted by \(A \cup B\), is the set that contains those elements that are either in \(A\) or in \(B\), or in both.
- Alternate: \(A \cup B = \{ x \mid x \in A \lor x \in B \}\).

**Example:**
- \(A = \{1,2,3,6\}\)
- \(B = \{2,4,6,9\}\)
- \(A \cup B = ?\)
Set operations

**Definition:** Let A and B be sets. The **union of A and B**, denoted by \( A \cup B \), is the set that contains those elements that are either in A or in B, or in both.

- Alternate: \( A \cup B = \{ x | x \in A \lor x \in B \} \).

**Example:**
- \( A = \{1,2,3,6\} \quad B = \{2,4,6,9\} \)
- \( A \cup B = \{1,2,3,4,6,9\} \)

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Set operations

**Definition:** Let A and B be sets. The **intersection of A and B**, denoted by \( A \cap B \), is the set that contains those elements that are in both A and B.

- Alternate: \( A \cap B = \{ x | x \in A \land x \in B \} \).

**Example:**
- \( A = \{1,2,3,6\} \quad B = \{2,4,6,9\} \)
- \( A \cap B = ? \)
Set operations

**Definition:** Let A and B be sets. The **intersection of A and B**, denoted by $A \cap B$, is the set that contains those elements that are in both A and B.

- Alternate: $A \cap B = \{ x \mid x \in A \land x \in B \}$.

**Example:**
- $A = \{1,2,3,6\}$  \hspace{1em} $B = \{2, 4, 6, 9\}$
- $A \cap B = \{2, 6\}$

Disjoint sets

**Definition:** Two sets are called **disjoint** if their intersection is empty.

- Alternate: A and B are disjoint if and only if $A \cap B = \emptyset$.

**Example:**
- $A = \{1,2,3,6\}$  \hspace{1em} $B = \{4,7,8\}$  Are these disjoint?
Disjoint sets

**Definition:** Two sets are called **disjoint** if their intersection is empty.

- Alternate: A and B are disjoint if and only if \( A \cap B = \emptyset \).

**Example:**
- \( A = \{1,2,3,6\} \quad B = \{4,7,8\} \) Are these disjoint?
- Yes.
- \( A \cap B = \emptyset \)

Cardinality of the set union

**Cardinality of the set union.**

- \( |A \cup B| = |A| + |B| - |A \cap B| \)

- Why this formula?
Cardinality of the set union

Cardinality of the set union.
• \(|A \cup B| = |A| + |B| - |A \cap B|\)

- Why this formula? Correct for an over-count.
- More general rule:
  - The principle of inclusion and exclusion.

Set difference

**Definition**: Let A and B be sets. The **difference of A and B**, denoted by \(A - B\), is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

- Alternate: \(A - B = \{ x \mid x \in A \land x \notin B \}\).

**Example**: \(A = \{1,2,3,5,7\}\  B = \{1,5,6,8\}\)
- \(A - B = \)?
Set difference

**Definition:** Let $A$ and $B$ be sets. The **difference of $A$ and $B$**, denoted by $A - B$, is the set containing those elements that are in $A$ but not in $B$. The difference of $A$ and $B$ is also called the complement of $B$ with respect to $A$.

- Alternate: $A - B = \{ x \mid x \in A \land x \not\in B \}$.

**Example:** $A = \{1,2,3,5,7\}$, $B = \{1,5,6,8\}$
- $A - B = \{2,3,7\}$

Complement of a set

**Definition:** Let $U$ be the **universal set**: the set of all objects under the consideration.

**Definition:** The **complement of the set $A$**, denoted by $\overline{A}$, is the complement of $A$ with respect to $U$.

- Alternate: $\overline{A} = \{ x \mid x \not\in A \}$

**Example:** $U=\{1,2,3,4,5,6,7,8\}$, $A=\{1,3,5,7\}$
- $\overline{A} =$ ?
Complement of a set

Definition: Let U be the universal set: the set of all objects under the consideration.

Definition: The complement of the set A, denoted by \( \overline{A} \), is the complement of A with respect to U.

• Alternate: \( \overline{A} = \{ x \mid x \notin A \} \)

Example: U={1,2,3,4,5,6,7,8} A ={1,3,5,7}
• \( \overline{A} = \{2,4,6,8\} \)

Set identities

Set Identities (analogous to logical equivalences)

• Identity
  – \( A \cup \emptyset = A \)
  – \( A \cap U = A \)

• Domination
  – \( A \cup U = U \)
  – \( A \cap \emptyset = \emptyset \)

• Idempotent
  – \( A \cup A = A \)
  – \( A \cap A = A \)
### Set identities

- **Double complement**
  - $\overline{\overline{A}} = A$

- **Commutative**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$

- **Associative**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$

- **Distributive**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

### Set identities

- **DeMorgan**
  - $\overline{A \cup B} = \overline{A} \cap \overline{B}$
  - $\overline{A \cap B} = \overline{A} \cup \overline{B}$

- **Absorption Laws**
  - $A \cup (A \cap B) = A$
  - $A \cap (A \cup B) = A$

- **Complement Laws**
  - $A \cup \overline{A} = U$
  - $A \cap \overline{A} = \emptyset$
Set identities

- Set identities can be proved using membership tables.
- List each combination of sets that an element can belong to. Then show that for each such a combination the element either belongs or does not belong to both sets in the identity.
- Prove: \((A \cap B) = \overline{A} \cup \overline{B}\)

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<th>(\overline{A})</th>
<th>(\overline{B})</th>
<th>(A \cap B)</th>
<th>(\overline{A} \cup \overline{B})</th>
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Generalized unions and intersections

**Definition:** The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

\[\bigcup_{i=1}^{n} A_i = \{A_1 \cup A_2 \cup \ldots \cup A_n\}\]

**Example:**

- Let \(A_i = \{1,2,\ldots,i\}\) \(i = 1,2,\ldots,n\)
- \[\bigcup_{i=1}^{n} A_i = \{1,2,\ldots,n\}\]
Generalized unions and intersections

**Definition:** The *intersection of a collection of sets* is the set that contains those elements that are members of all sets in the collection.

\[ \bigcap_{i=1}^{n} A_i = \{ A_1 \cap A_2 \cap ... \cap A_n \} \]

**Example:**
- Let \( A_i = \{1, 2, ..., i\} \quad i = 1, 2, ..., n \)

\[ \bigcap_{i=1}^{n} A_i = \{1\} \]

Computer representation of sets

**Idea:** Assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is present otherwise use 0.

**Example:**

All possible elements: \( U = \{1, 2, 3, 4, 5\} \)
- Assume \( A = \{2, 5\} \)
  - Computer representation: \( A = 01001 \)
- Assume \( B = \{1, 5\} \)
  - Computer representation: \( B = 10001 \)
Computer representation of sets

Example:
• A = 01001
• B = 10001

• The **union** is modeled with a bitwise **or**
• A ∨ B = 11001
• The **intersection** is modeled with a bitwise **and**
• A ∧ B = 00001
• The **complement** is modeled …?