Sets and set operations

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Set

- **Definition**: A set is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set. (Cantor's naive definition)

- **Examples**:
  - **Vowels in the English alphabet**
    
    \[ V = \{ \text{a, e, i, o, u} \} \]
  
  - **First seven prime numbers**.
    
    \[ X = \{ 2, 3, 5, 7, 11, 13, 17 \} \]
Representing sets

**Representing a set:**
1) Listing the members.
2) Definition by property, using set builder notation 
\( \{x | x \text{ has property } P \} \).

**Example:**
- Even integers between 50 and 63.
  1) \( E = \{50, 52, 54, 56, 58, 60, 62\} \)
  2) \( E = \{x | 50 \leq x < 63, \text{x is an even integer}\} \)

If enumeration of the members is hard we often use ellipses.

**Example:** a set of integers between 1 and 100
- \( A = \{1,2,3, \ldots, 100\} \)

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**Important sets in discrete math**

- **Natural numbers:**
  - \( N = \{0,1,2,3, \ldots\} \)

- **Integers**
  - \( Z = \{\ldots, -2,-1,0,1,2, \ldots\} \)

- **Positive integers**
  - \( Z^+ = \{1,2,3, \ldots\} \)

- **Rational numbers**
  - \( Q = \{p/q | p \in Z, q \in Z, q \neq 0\} \)

- **Real numbers**
  - \( R \)
Special sets

- **Special sets:**
  - The universal set is denoted by \( U \): the set of all objects under the consideration.
  - The empty set is denoted as \( \emptyset \) or \{ \}.

Venn diagrams

- A set can be visualized using **Venn Diagrams**:
  - \( V=\{ A, B, C \} \)
A Subset

- **Definition:** A set \( A \) is said to be a **subset** of \( B \) if and only if every element of \( A \) is also an element of \( B \). We use \( A \subseteq B \) to indicate **\( A \) is a subset of \( B \)**.

- Alternate way to define \( A \) is a subset of \( B \):
  \[
  \forall x \ (x \in A) \rightarrow (x \in B)
  \]

Empty set/Subset properties

**Theorem** \( \emptyset \subseteq S \)

- **Empty set is a subset of any set.**

**Proof:**
- Recall the definition of a subset: all elements of a set \( A \) must be also elements of \( B \): \( \forall x \ (x \in A) \rightarrow (x \in B) \).
- We must show the following implication holds for any \( S \)
  \[
  \forall x \ (x \in \emptyset) \rightarrow (x \in S)
  \]
- ?
Empty set/Subset properties

**Theorem** $\emptyset \subseteq S$

- Empty set is a subset of any set.

**Proof:**

- Recall the definition of a subset: all elements of a set $A$ must be also elements of $B$: $\forall x (x \in A \implies x \in B)$.
- We must show the following implication holds for any $S$

  $\forall x (x \in \emptyset \implies x \in S)$

- Since the empty set does not contain any element, $x \in \emptyset$ is always **False**
- Then the implication is **always True**.

End of proof

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Subset properties

**Theorem:** $S \subseteq S$

- Any set $S$ is a subset of itself

**Proof:**

- the definition of a subset says: all elements of a set $A$ must be also elements of $B$: $\forall x (x \in A \implies x \in B)$.
- Applying this to $S$ we get:

  $\forall x (x \in S \implies x \in S)$ …
Subset properties

**Theorem:** $S \subseteq S$

- Any set $S$ is a subset of itself

**Proof:**

- the definition of a subset says: all elements of a set $A$ must be also elements of $B$: $\forall x (x \in A \rightarrow x \in B)$.
- Applying this to $S$ we get:
- $\forall x (x \in S \rightarrow x \in S)$ which is trivially True
- End of proof

**Note on equivalence:**

- Two sets are equal if each is a subset of the other set.

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A proper subset

**Definition:** A set $A$ is said to be a **proper subset** of $B$ if and only if $A \subseteq B$ and $A \neq B$. We denote that $A$ is a proper subset of $B$ with the notation $A \subset B$. 

![Diagram of sets A and B within set U](image)
**A proper subset**

**Definition:** A set $A$ is said to be a proper subset of $B$ if and only if $A \subseteq B$ and $A \neq B$. We denote that $A$ is a proper subset of $B$ with the notation $A \subset B$.

Example: $A = \{1,2,3\}$ $B = \{1,2,3,4,5\}$
Is: $A \subset B$? Yes.
Cardinality

**Definition:** Let $S$ be a set. If there are exactly $n$ distinct elements in $S$, where $n$ is a nonnegative integer, we say $S$ is a finite set and that $n$ is the **cardinality of** $S$. The cardinality of $S$ is denoted by $|S|$.

**Examples:**
- $V=\{1, 2, 3, 4, 5\}$
  
  $|V| = ?$

- $A=\{1, 2, 3, 4, \ldots, 20\}$
  
  $|A| = ?$
Cardinality

**Definition:** Let $S$ be a set. If there are exactly $n$ distinct elements in $S$, where $n$ is a nonnegative integer, we say $S$ is a finite set and that $n$ is the **cardinality of $S$**. The cardinality of $S$ is denoted by $|S|$.

**Examples:**
- $V = \{1, 2, 3, 4, 5\}$  
  $|V| = 5$
- $A = \{1, 2, 3, 4, ..., 20\}$  
  $|A| = 20$
- $|\emptyset| = 0$
Infinite set

**Definition:** A set is **infinite** if it is not finite.

**Examples:**
- The set of natural numbers is an infinite set.
  \[ N = \{0, 1, 2, 3, \ldots \} \]
- The set of reals is an infinite set.

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Power set

**Definition:** Given a set \( S \), the **power set** of \( S \) is the set of all subsets of \( S \). The power set is denoted by \( P(S) \).

**Examples:**
- Assume an empty set \( \emptyset \)
- What is the power set of \( \emptyset \)? \( P(\emptyset) = ? \)
Power set

Definition: Given a set $S$, the power set of $S$ is the set of all subsets of $S$. The power set is denoted by $P(S)$.

Examples:
- Assume an empty set $\emptyset$
- What is the power set of $\emptyset$? $P(\emptyset) = \{\emptyset\}$
- What is the cardinality of $P(\emptyset)$? $|P(\emptyset)| = 1$.
- Assume set $\{1\}$
- $P(\{1\}) = ?$
**Power set**

**Definition:** Given a set S, the power set of S is the set of all subsets of S. The power set is denoted by $P(S)$.

**Examples:**
- Assume an empty set $\emptyset$.
- What is the power set of $\emptyset$? $P(\emptyset) = \{ \emptyset \}$
- What is the cardinality of $P(\emptyset)$? $|P(\emptyset)| = 1$.

- Assume set $\{1\}$
- $P(\{1\}) = \{ \emptyset, \{1\} \}$
- $|P(\{1\})| = 2$
Power set

- \( P(\{1\}) = \{\emptyset, \{1\}\} \)
- \(|P(\{1\})| = 2\)

- Assume \(\{1,2\}\)
- \(P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}\)
- \(|P(\{1,2\})| = ?\)
Power set

- $P(\{1\}) = \{\emptyset, \{1\}\}$
- $|P(\{1\})| = 2$

- Assume $\{1,2\}$
- $P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
- $|P(\{1,2\})| = 4$

- Assume $\{1,2,3\}$
- $P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- $|P(\{1,2,3\})| = ?$
Power set

- $P(\{1\}) = \{\emptyset, \{1\}\}$
- $|P(\{1\})| = 2$

- Assume $\{1,2\}$
- $P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
- $|P(\{1,2\})| = 4$

- Assume $\{1,2,3\}$
- $P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- $|P(\{1,2,3\})| = 8$

- If $S$ is a set with $|S| = n$ then $|P(S)| = 2^n$.

N-tuple

- Sets are used to represent unordered collections.
- Ordered-$n$ tuples are used to represent ordered collections.

**Definition:** An ordered n-tuple $(x_1, x_2, ..., x_N)$ is the ordered collection that has $x_1$ as its first element, $x_2$ as its second element, ..., and $x_N$ as its $N$-th element, $N \geq 2$.

**Example:**

- Coordinates of a point in the 2-D plane $(12, 16)$
**Cartesian product**

**Definition:** Let $S$ and $T$ be sets. The **Cartesian product of $S$ and $T$**, denoted by $S \times T$, is the set of all ordered pairs $(s,t)$, where $s \in S$ and $t \in T$. Hence,

$$S \times T = \{ (s,t) \mid s \in S \land t \in T \}.$$  

**Examples:**

- $S = \{1,2\}$ and $T = \{a,b,c\}$
- $S \times T = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$
- $T \times S = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$
- Note: $S \times T \neq T \times S$ !!!!