CS 441 Discrete Mathematics for CS
Lecture 10

Sets and set operations

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Set

- **Definition**: A **set** is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set. (Cantor's naive definition)

- **Examples:**
  - **Vowels in the English alphabet**
    V = { a, e, i, o, u }
  - **First seven prime numbers**.
    X = { 2, 3, 5, 7, 11, 13, 17 }

Representing sets

Representing a set:

1) Listing the members.

2) Definition by property, using set builder notation

{x| x has property P}.

Example:

- Even integers between 50 and 63.

1) E = {50, 52, 54, 56, 58, 60, 62}

2) E = {x| 50 <= x < 63, x is an even integer}

If enumeration of the members is hard we often use ellipses.

Example: a set of integers between 1 and 100

- **A= {1,2,3 …, 100}**

Important sets in discrete math

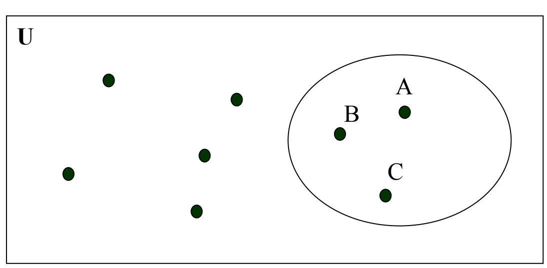
- **Natural numbers:**
  - **N** = {0,1,2,3, …}
- **Integers**
  - **Z** = {…, -2,-1,0,1,2, …}
- **Positive integers**
  - $\mathbf{Z}^+$ = {1,2, 3.…}
- **Rational numbers**
  - **Q** = $\{p/q \mid p \in Z, q \in Z, q \neq 0\}$
- **Real numbers**
  - **R**

Special sets

- **Special sets:**
  - **The <u>universal set</u>** is denoted by **U:** the set of all objects under the consideration.
  - **The <u>empty set</u>** is denoted as ∅ **or { }.**

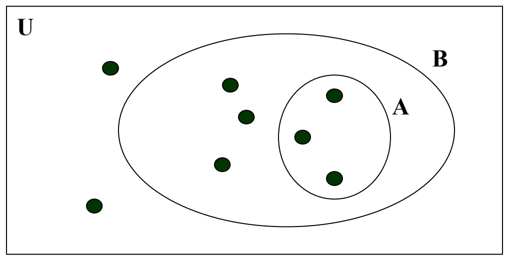
Venn diagrams

- A set can be visualized using **Venn Diagrams**:
  - V={ A, B, C }



A Subset

- **Definition**: A **set** A is said to be a **subset** of B **if and only if** every element of A is also an element of B. We use $A \subseteq B$ to indicate **A is a subset of B**.



- Alternate way to define A is a subset of B:
  $\forall x\ (x \in A) \rightarrow (x \in B)$

Empty set/Subset properties

Theorem $\emptyset \subseteq S$

- **Empty set is a subset of any set.**

Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x\ (x \in A) \rightarrow (x \in B)$.
- We must show the following implication holds for any S
  $\forall x\ (x \in \emptyset) \rightarrow (x \in S)$
- ?

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Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x \, (x \in A \rightarrow x \in B)$.
- We must show the following implication holds for any S
  $\forall x \, (x \in \emptyset \rightarrow x \in S)$
- Since the empty set does not contain any element, $x \in \emptyset$ is **always False**
- Then the implication is **always True.**

End of proof

Subset properties

Theorem: $S \subseteq S$

- **Any set S is a subset of itself**

Proof:

- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x \, (x \in A) \rightarrow (x \in B)$.
- Applying this to S we get:
- $\forall x \, (x \in S) \rightarrow (x \in S)$ …

Subset properties

Theorem: **$S \subseteq S$**

- **Any set S is a subset of itself**

Proof:

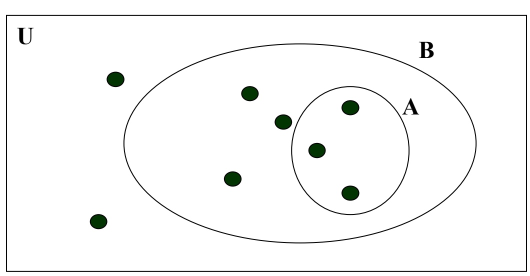
- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.
- Applying this to S we get:
- $\forall x\ (x \in S \rightarrow x \in S)$ which is trivially **True**
- End of proof

Note on equivalence:

- Two sets are equal if each is a subset of the other set.

A proper subset

Definition: A **set A** is said to be a **proper subset** of B if and only if **$A \subseteq B$** and **$A \neq B$**. We denote that A is a proper subset of B with the notation $A \subset B$.



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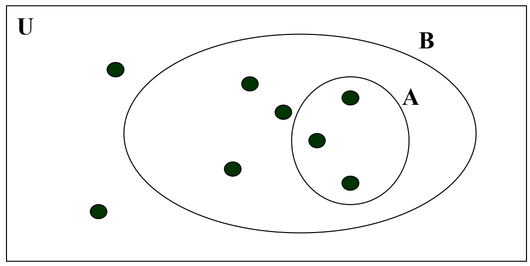

Example: A={1,2,3} B ={1,2,3,4,5}

Is: $A \subset B$?

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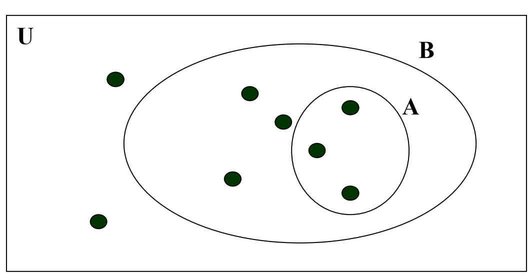

Example: A={1,2,3} B ={1,2,3,4,5}

Is: $A \subset B$? Yes.

Cardinality

Definition: Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say S is a finite set and that n is the **cardinality of S**. The cardinality of S is denoted by **| S |.**

Examples:

- **V={1 2 3 4 5}**
  | V | = ?

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Examples:

- **V={1 2 3 4 5}**
  | V | = 5

- **A={1,2,3,4, …, 20}**
  |A| =?

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Examples:

- **V={1 2 3 4 5}**
  | V | = 5
- **A={1,2,3,4, …, 20}**
  |A| =20
- **| ∅ | = ?**

Cardinality

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Examples:

- **V={1 2 3 4 5}**
  | V | = 5
- **A={1,2,3,4, …, 20}**
  |A| =20
- **| ∅ | = 0**

Infinite set

Definition: A set is **infinite** if it is not finite.

Examples:

- The set of natural numbers is an infinite set.
- N = {0, 1, 2, 3, ... }
- The set of reals is an infinite set.

Power set

Definition: Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set $\emptyset$
- What is the power set of $\emptyset$? $P(\emptyset) = ?$

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Examples:

- Assume an empty set $\emptyset$
- What is the power set of $\emptyset$? $P(\emptyset) = \{ \emptyset \}$
- What is the cardinality of $P(\emptyset)$? $| P(\emptyset) | = ?$

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Examples:

- Assume an empty set $\emptyset$
- What is the power set of $\emptyset$? $P(\emptyset) = \{ \emptyset \}$
- What is the cardinality of $P(\emptyset)$? $| P(\emptyset) | = 1$.

- Assume set $\{1\}$
- $P(\{1\}) = ?$

Power set

Definition: Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set ∅
- What is the power set of ∅ ? P(∅) = { ∅ }
- What is the cardinality of P(∅) ? | P(∅) | = 1.

- Assume set {1}
- P({1}) = { ∅, {1} }
- |P({1})| = ?

Power set

Definition: Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set ∅
- What is the power set of ∅ ? P(∅) = { ∅ }
- What is the cardinality of P(∅) ? | P(∅) | = 1.

- Assume set {1}
- P({1}) = { ∅, {1} }
- |P({1})| = 2

Power set

- P({1}) = { ∅, {1} }
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- Assume {1,2}
- P({1,2}) =

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- Assume {1,2}
- P({1,2}) = { ∅, {1}, {2}, {1,2} }
- |P({1,2})| =?

Power set

- P({1}) = { ∅, {1} }
- |P({1})| = 2

- Assume {1,2}
- P({1,2}) = { ∅, {1}, {2}, {1,2} }
- |P({1,2})| =4

- Assume {1,2,3}
- P({1,2,3}) = ?

Power set

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- Assume {1,2,3}
- P({1,2,3}) = {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }
- |P({1,2,3} | = ?

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- Assume {1,2,3}
- P({1,2,3}) = {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }
- |P({1,2,3} | = 8

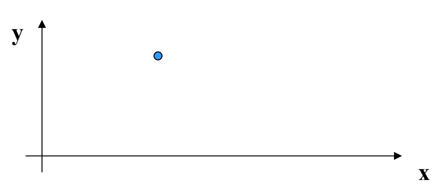
- **If S is a set with |S| = n then | P(S) | = 2^n..**

N-tuple

- Sets are used to represent unordered collections.
- Ordered-n tuples are used to represent ordered collection.

Definition: An **ordered n-tuple** (x1, x2, ..., xN) is the ordered collection that has x1 as its first element, x2 as its second element, ..., and xN as its N-th element, N ≥ 2.

Example:



- Coordinates of a point in the 2-D plane (12, 16)

Cartesian product

Definition: Let S and T be sets. The **Cartesian product of S and T**, denoted by **S x T,** is the set of all ordered pairs (s,t), where s $\in$ S and t $\in$ T. Hence,

- $S \times T = \{ (s,t) \mid s \in S \wedge t \in T\}$.

Examples:

- S = {1,2} and T = {a,b,c}
- S x T = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }
- T x S = { (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) }
- Note: S x T $\neq$ T x S **!!!!**