CS 2750 Machine Learning
Lecture 8

Classification learning II

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Classification

- **Data:** $D = \{d_1, d_2, .., d_n\}$

  $d_i = < \mathbf{x}_i, y_i >$

  – y_i **represents a discrete class value**
- **Goal: learn** $f : X \rightarrow Y$

- **Binary classification**
  – **A special case when** $Y \in \{0,1\}$

- **First step:**
  – we need to devise a model of the function f

Discriminant functions

- A common way to represent a **classifier is by using**
  - **Discriminant functions**
- **Works for both the binary and multi-way classification**
- **Idea:**
  - For every class $i = 0,1, \ldots k$ define a function $g_i(\mathbf{x})$ mapping $X \rightarrow \Re$
  - When the decision on input **x** should be made choose the class with the highest value of $g_i(\mathbf{x})$

$$y^* = \arg\max_i \; g_i(\mathbf{x})$$

Discriminant functions

- **Discriminant functions depend on parameters**

$$g_i(\mathbf{x}) \quad \sim g_i(\mathbf{x}, \mathbf{w})$$

- **Logistic regression model (for 2 classes)**

$$g_1(\mathbf{x}, \mathbf{w}) = g(\mathbf{w}^T \mathbf{x}) \qquad g_0(\mathbf{x}, \mathbf{w}) = 1 - g(\mathbf{w}^T \mathbf{x})$$

$g(z) = 1/(1 + e^{-z})$ - is a logistic function

$g_1(\mathbf{x}, \mathbf{w}) \sim p(y = 1 \mid \mathbf{x}, \mathbf{w})$ - Models directly class posterior

- **Generative probabilistic model (QDA)**

$$g_1(x, \Theta) = p(y = 1 \mid \mathbf{x}, \Theta) = \frac{p(\mathbf{x} \mid \mu_1, \mathbf{\Sigma}_1) p(y = 1)}{p(\mathbf{x} \mid \mu_0, \mathbf{\Sigma}_0) p(y = 0) + p(\mathbf{x} \mid \mu_1, \mathbf{\Sigma}_1) p(y = 1)}$$

$$g_0(x, \Theta) = p(y = 0 \mid \mathbf{x}, \Theta) = 1 - g_1(x, \Theta)$$

Discriminative vs generative models

- What is the difference in between discriminative and generative models and learning?
- **Discriminative:** **learns directly the discriminant functions and their parameters**
- Define the loss function using $g_1(\mathbf{x}, \mathbf{w})$ and $\mathbf{y}$

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} Loss(g_1(\mathbf{x}, \mathbf{w}), \mathbf{y})$$

- **Generative probabilistic:** **learns the joint probability distribution p(x,y) and its parameters, discriminant functions are derived from the joint.**

E.g: ML estimate of the parameters of the joint model

$$\Theta^* = \arg\max_{\Theta} p(D \mid \Theta) = \arg\max_{\Theta} \prod_{i=1}^{n} p(\mathbf{x}_i, y_i \mid \Theta)$$

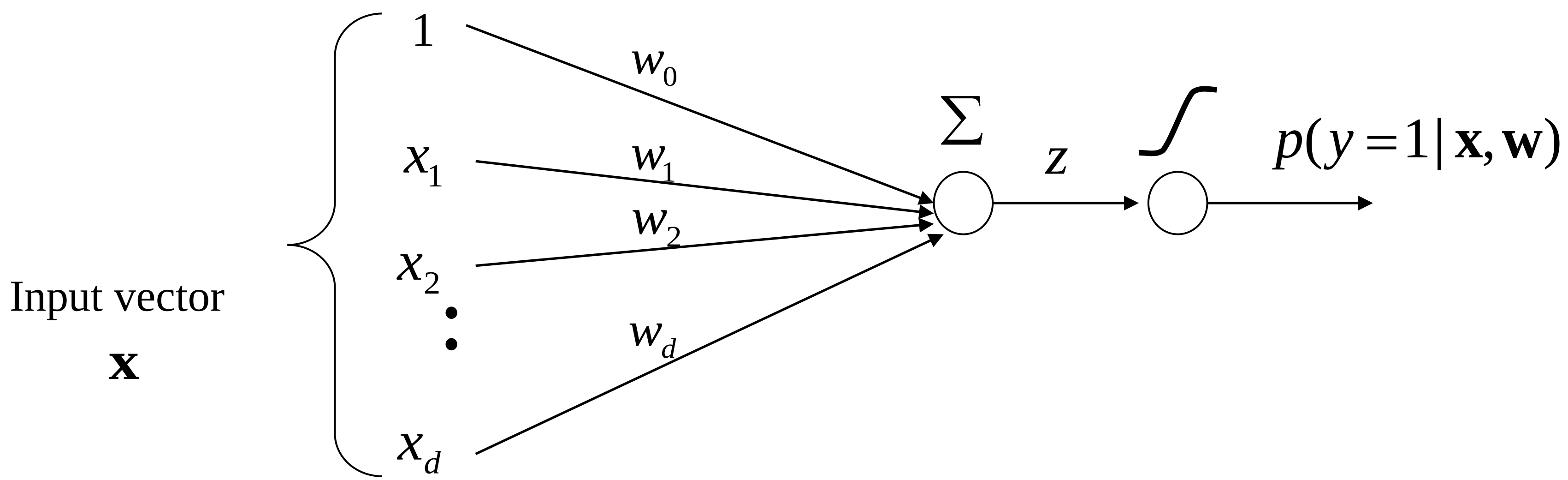
Logistic regression model

- **Discriminant functions:**

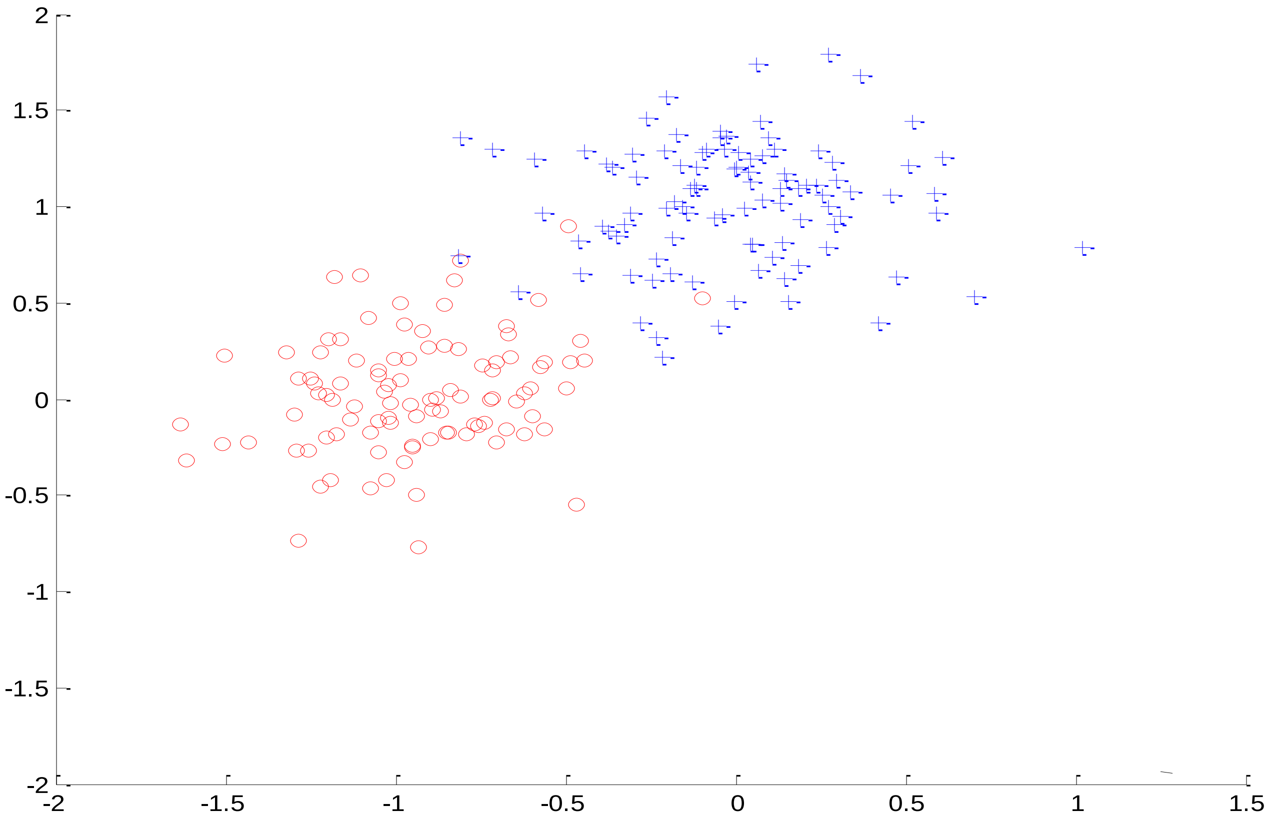
$$g_1(\mathbf{x}) = g(\mathbf{w}^T\mathbf{x}) \qquad\qquad g_0(\mathbf{x}) = 1 - g(\mathbf{w}^T\mathbf{x})$$

- **Values of discriminant functions vary in [0,1]**
  - **Probabilistic interpretation**

$$g_1(\mathbf{x},\mathbf{w}) = p(y=1 \mid \mathbf{w},\mathbf{x}) = g(\mathbf{w}^T\mathbf{x})$$



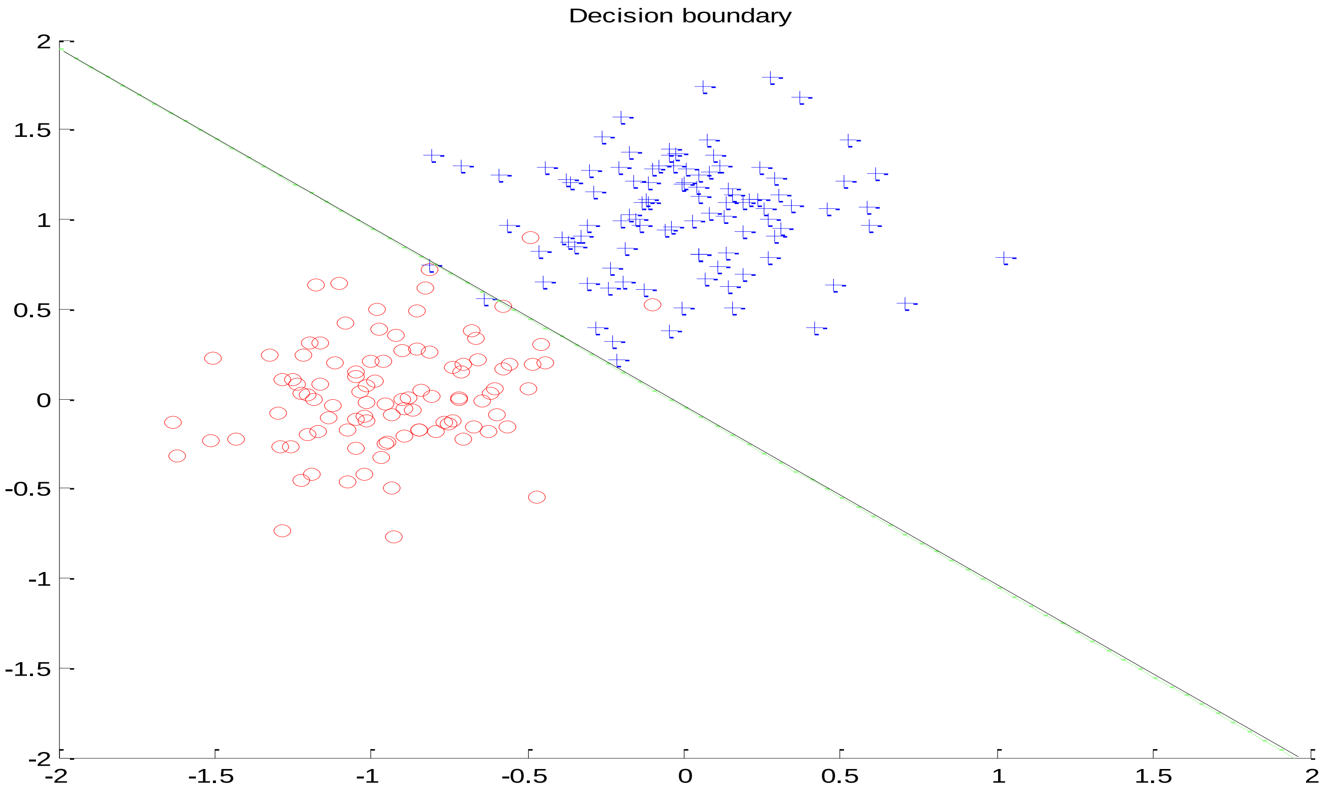
Binary classification example



Logistic regression model. Decision boundary

- **LR defines a linear decision boundary**

Example: 2 classes (blue and red points)



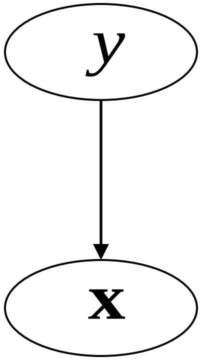
Generative approach to classification

Idea:

1. **Represent and learn the distribution** $p(\mathbf{x}, y)$
2. **Use it to define probabilistic discriminant functions**

E.g. $g_o(\mathbf{x}) = p(y = 0 \mid \mathbf{x}) \qquad g_1(\mathbf{x}) = p(y = 1 \mid \mathbf{x})$

Typical model $\quad p(\mathbf{x}, y) = p(\mathbf{x} \mid y) p(y)$



- $p(\mathbf{x} \mid y)$ = **Class-conditional distributions (densities)**

  binary classification: two class-conditional distributions

  $$p(\mathbf{x} \mid y = 0) \qquad\qquad p(\mathbf{x} \mid y = 1)$$

- $p(y)$ = **Priors on classes** - probability of class y

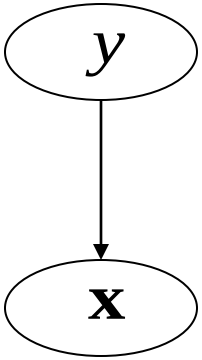
  binary classification: Bernoulli distribution

  $$p(y = 0) + p(y = 1) = 1$$

Quadratic discriminant analysis (QDA)

Model:

- **Class-conditional distributions**
  - **multivariate normal distributions**



$$\mathbf{x} \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \quad \text{for} \quad y = 0$$

$$\mathbf{x} \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) \quad \text{for} \quad y = 1$$

Multivariate normal $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]$$

- **Priors on classes (class 0,1)** $y \sim Bernoulli$
  - **Bernoulli distribution**

$$p(y, \theta) = \theta^y (1 - \theta)^{1-y} \qquad y \in \{0,1\}$$

Learning of parameters of the QDA model

Density estimation in statistics

- We see examples – we do not know the parameters of Gaussians (class-conditional densities)

$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]$$

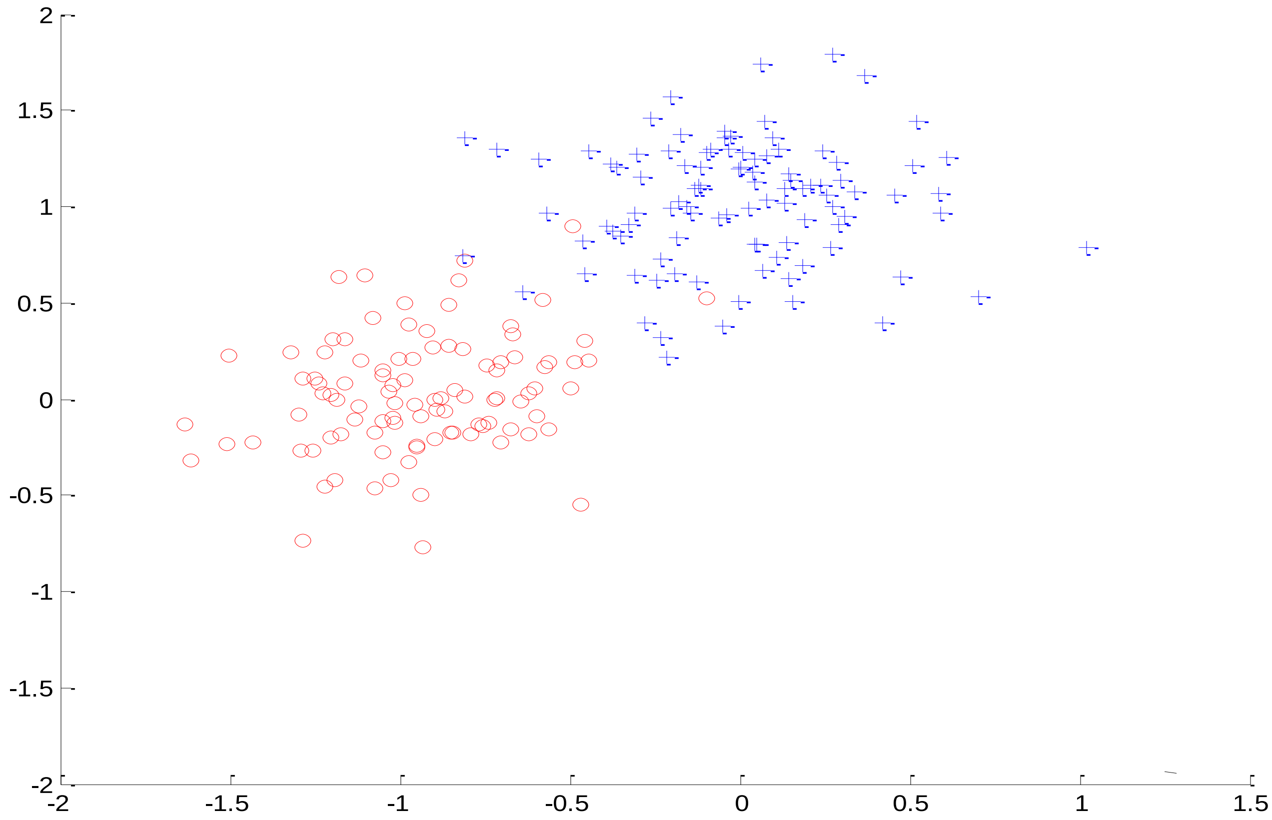
- **ML estimate of parameters** of a multivariate normal $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for a set of n examples of $\mathbf{x}$

  Optimize log-likelihood: $l(D, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \log \prod_{i=1}^{n} p(\mathbf{x}_i \mid \boldsymbol{\mu}, \boldsymbol{\Sigma})$

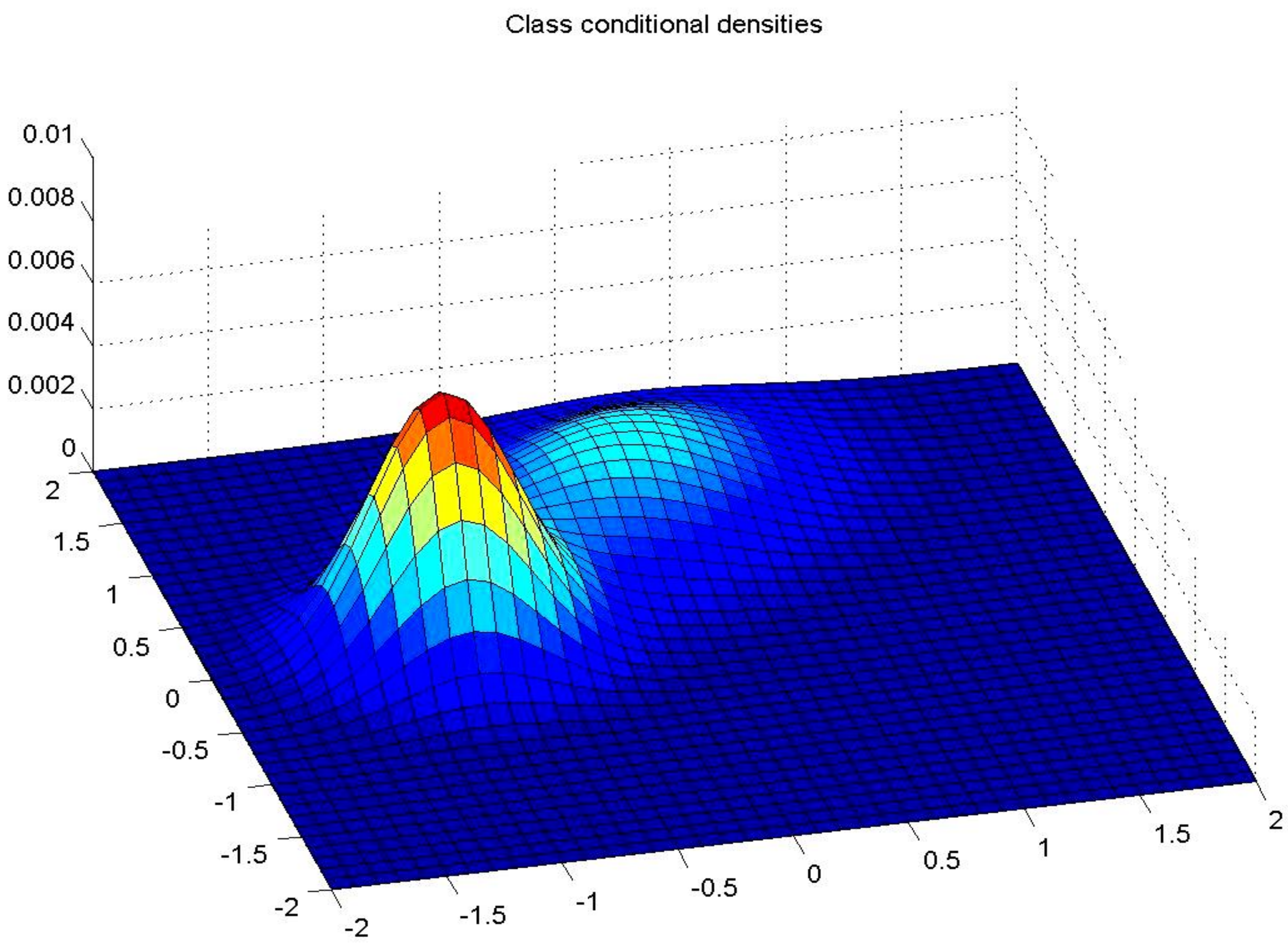
$$\hat{\boldsymbol{\mu}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \qquad \hat{\boldsymbol{\Sigma}} = \frac{1}{n} \sum_{i=1}^{n} (\mathbf{x}_\mathbf{i} - \hat{\boldsymbol{\mu}})(\mathbf{x}_\mathbf{i} - \hat{\boldsymbol{\mu}})^T$$

- How about **class priors**?

QDA



2 Gaussian class-conditional densities



QDA: Making class decision

Basically we need to design discriminant functions

Two possible choices:

- **Likelihood of data** – choose the class (Gaussian) that explains the input data ($\mathbf{x}$) better (likelihood of the data)

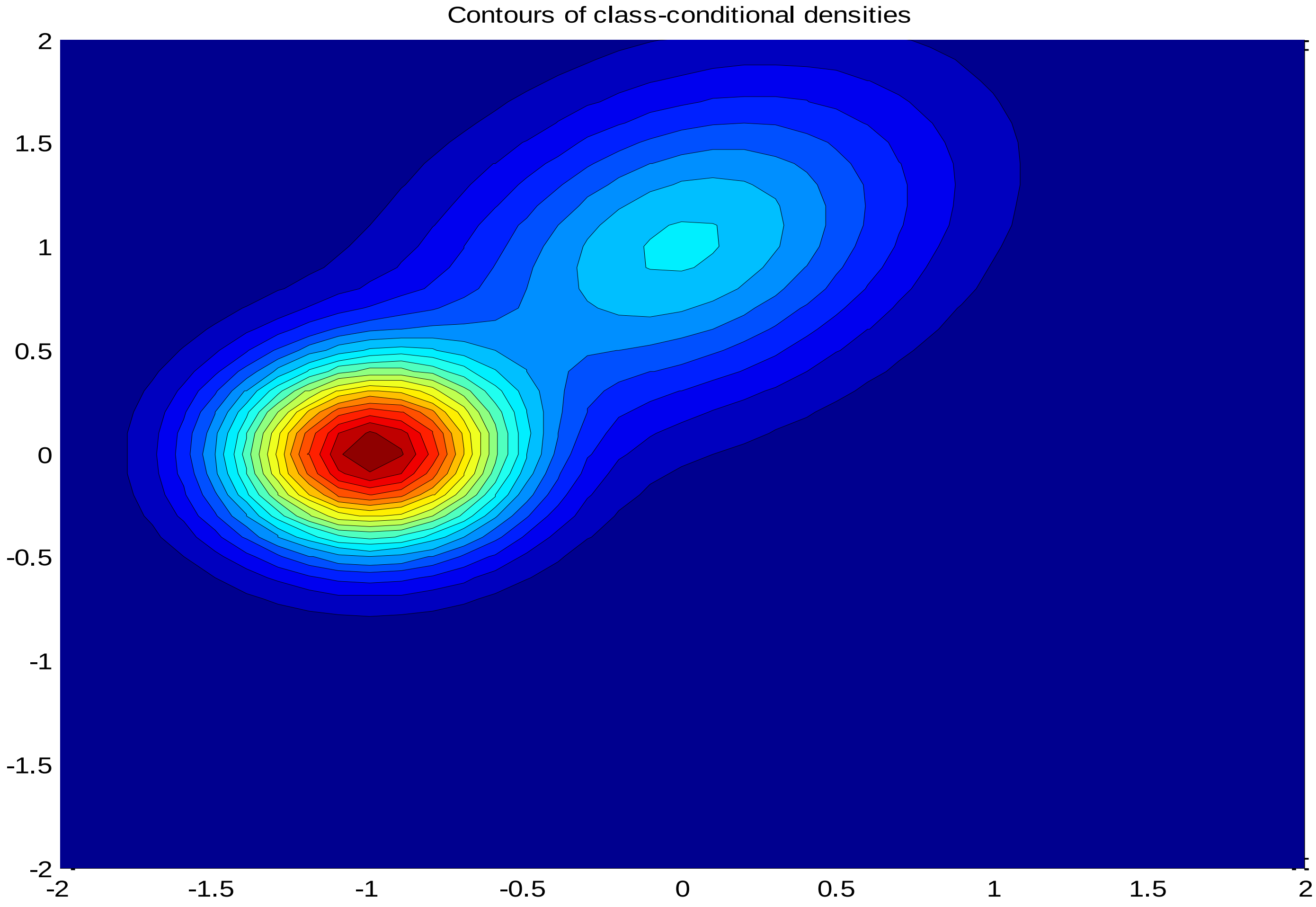
$$\underbrace{p(\mathbf{x} \mid \mu_1, \mathbf{\Sigma}_1)}_{g_1(\mathbf{x})} > \underbrace{p(\mathbf{x} \mid \mu_0, \mathbf{\Sigma}_0)}_{g_0(\mathbf{x})} \Rightarrow \begin{array}{l} \text{then } \; y=1 \\ \text{else } \; y=0 \end{array}$$

- **Posterior of a class** – choose the class with better posterior probability

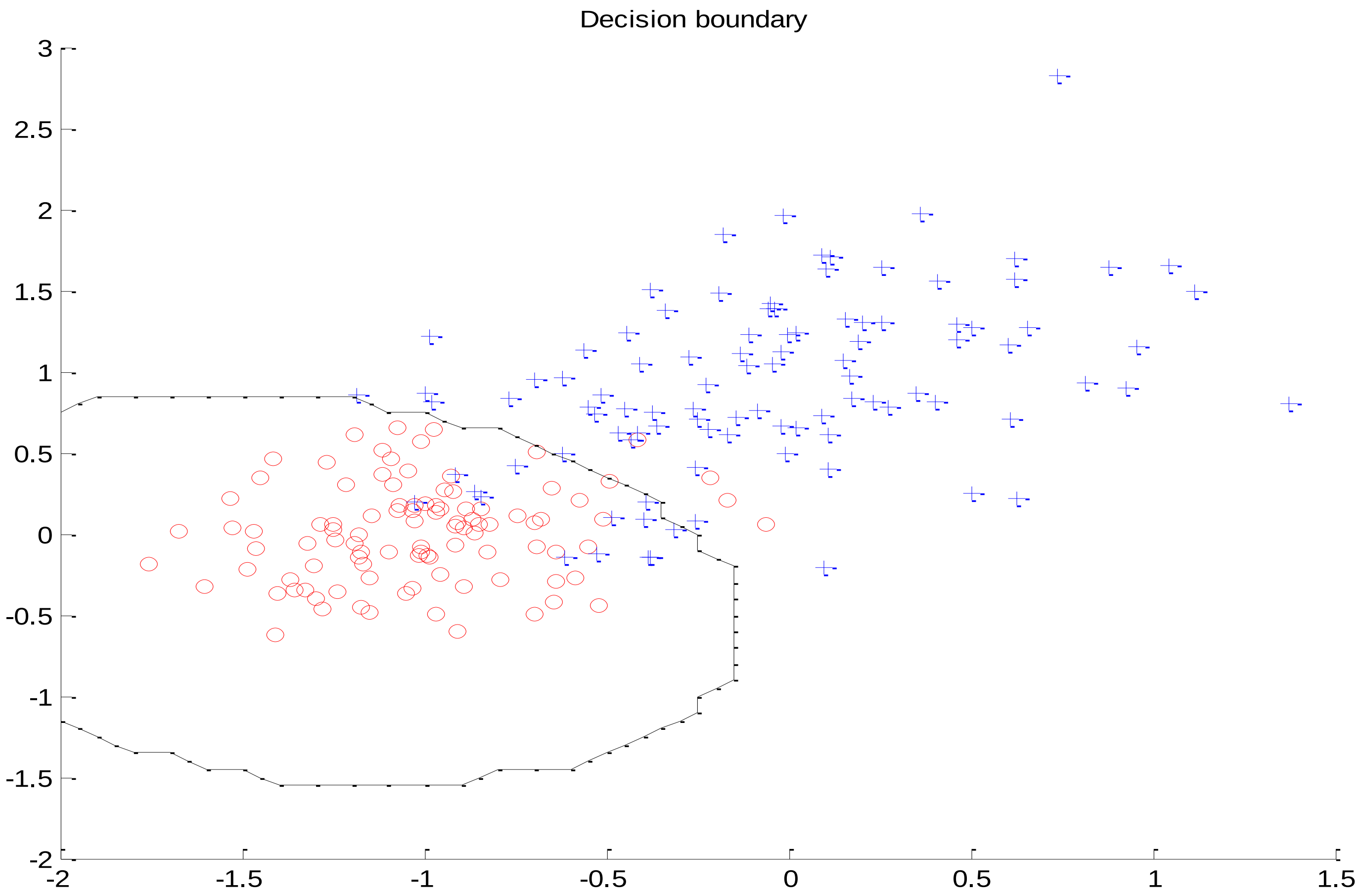
$$p(y=1 \mid \mathbf{x}) > p(y=0 \mid \mathbf{x}) \qquad \begin{array}{l} \text{then } \; y=1 \\ \text{else } \; y=0 \end{array}$$

$$p(y=1 \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid \mu_1, \mathbf{\Sigma}_1)\, p(y=1)}{p(\mathbf{x} \mid \mu_0, \mathbf{\Sigma}_0)\, p(y=0) + p(\mathbf{x} \mid \mu_1, \mathbf{\Sigma}_1)\, p(y=1)}$$

QDA: Quadratic decision boundary



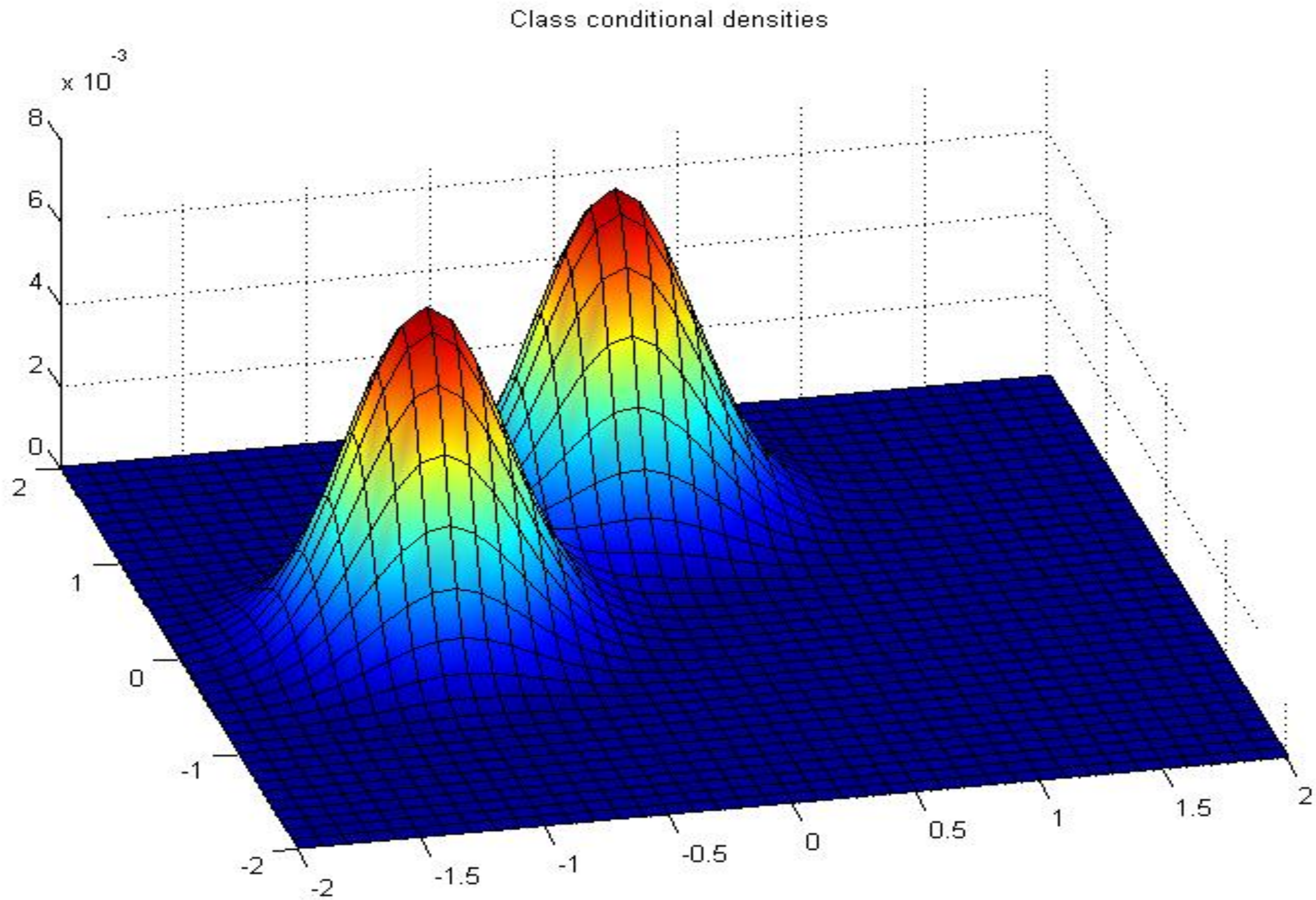
QDA: Quadratic decision boundary



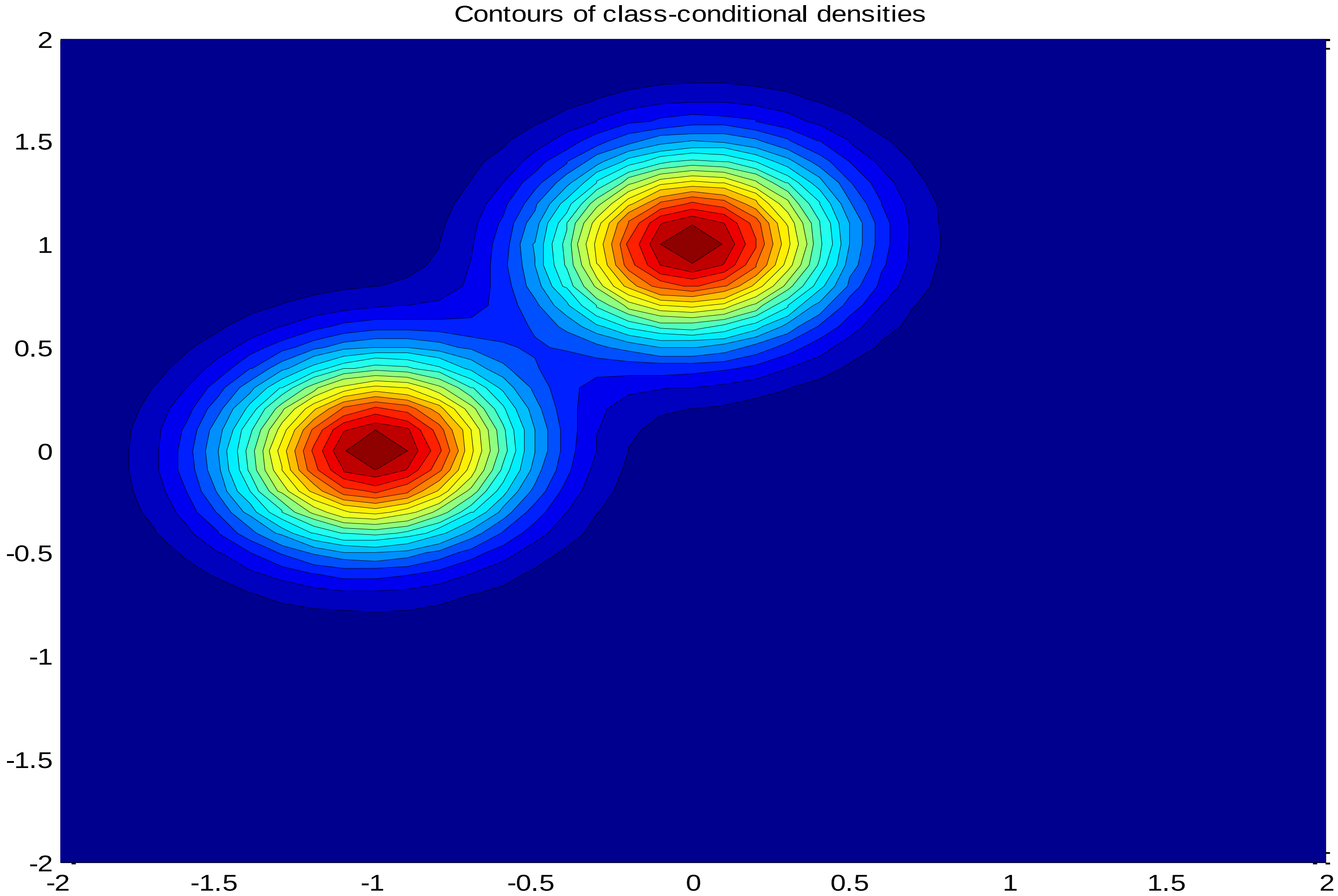
Linear discriminant analysis (LDA)

- When covariances are the same

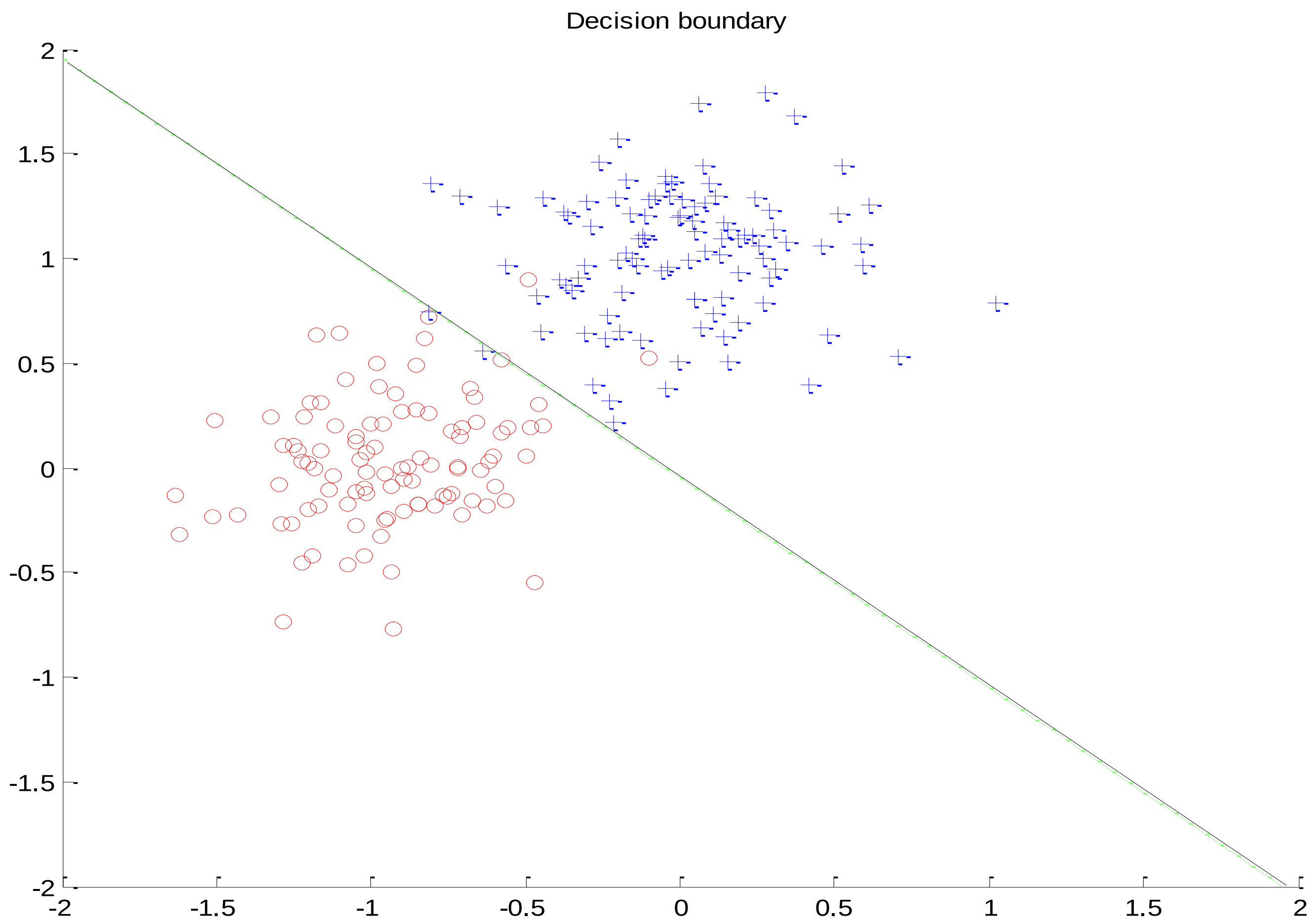
$$\mathbf{x} \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}),\ y = 0$$
$$\mathbf{x} \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}),\ y = 1$$



LDA: Linear decision boundary



LDA: linear decision boundary



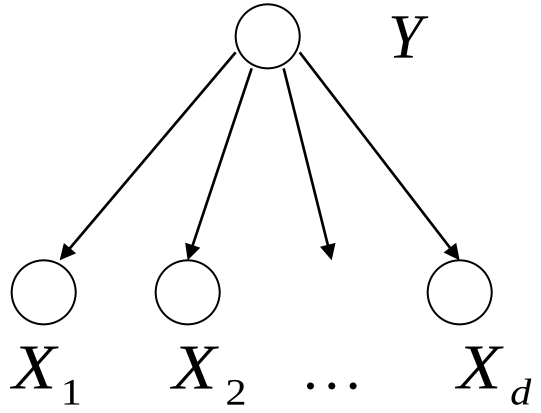
Naïve Bayes classifier

- **A generative classifier model with an additional simplifying assumption:**
  - **All input attributes are conditionally independent of each other given the class.**

So we have:

$$p(\mathbf{x}, y) = p(\mathbf{x} \mid y)\, p(y)$$

$$p(\mathbf{x} \mid y) = \prod_{i=1}^{d} p(x_i \mid y)$$



Learning parameters of the model

Much simpler density estimation problems

- We need to learn:

$$p(\mathbf{x} \mid y = 0) \quad \text{and} \quad p(\mathbf{x} \mid y = 1) \quad \text{and} \quad p(y)$$

- Because of the assumption of the conditional independence we need to learn:

for every variable i: $p(x_i \mid y = 0)$ and $p(x_i \mid y = 1)$

- **Much easier if the number of input attributes is large**
- **Also, the model gives us a flexibility to represent input attributes different of different forms !!!**
- E.g. one attribute can be modeled using the Bernoulli, the other as Gaussian density, or as a Poisson distribution

Making a class decision for the Naïve Bayes

Discriminant functions

- **Likelihood of data** – choose the class that explains the input data ($\mathbf{x}$) better (likelihood of the data)

$$\underbrace{\prod_{i=1}^{d} p(x_i \mid \Theta_{1,i})}_{g_1(\mathbf{x})} > \underbrace{\prod_{i=1}^{d} p(x_i \mid \Theta_{2,i})}_{g_0(\mathbf{x})} \Rightarrow \begin{array}{l} \text{then } \; y=1 \\ \text{else } \; y=0 \end{array}$$

- **Posterior of a class** – choose the class with better posterior probability $p(y=1 \mid \mathbf{x}) > p(y=0 \mid \mathbf{x})$

$$\begin{array}{l} \text{then } \; y=1 \\ \text{else } \; y=0 \end{array}$$

$$p(y=1 \mid \mathbf{x}) = \frac{\left(\prod_{i=1}^{d} p(x_i \mid \Theta_{1,i})\right) p(y=1)}{\left(\prod_{i=1}^{d} p(x_i \mid \Theta_{1,i})\right) p(y=0) + \left(\prod_{i=1}^{d} p(x_i \mid \Theta_{2,i})\right) p(y=1)}$$

Back to logistic regression

- **Two models with linear decision boundaries:**
  - **Logistic regression**
  - **Generative model with 2 Gaussians with the same covariance matrices**

$$x \sim N(\mu_0, \Sigma) \quad \text{for} \quad y = 0$$
$$x \sim N(\mu_1, \Sigma) \quad \text{for} \quad y = 1$$

- **Two models are related !!!**
  - When we have **2 Gaussians with the same covariance matrix** the probability of y given **x** has the form of a logistic regression model **!!!**

$$\boxed{p(y = 1 \mid \mathbf{x}, \boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \boldsymbol{\Sigma}) = g(\mathbf{w}^T \mathbf{x})}$$

When is the logistic regression model correct?

- **Members of the exponential family can be often more naturally described as**

$$f(\mathbf{x} \mid \boldsymbol{\theta}, \boldsymbol{\varphi}) = h(x, \boldsymbol{\varphi}) \exp\left\{ \frac{\boldsymbol{\theta}^T \mathbf{x} - A(\boldsymbol{\theta})}{a(\boldsymbol{\varphi})} \right\}$$

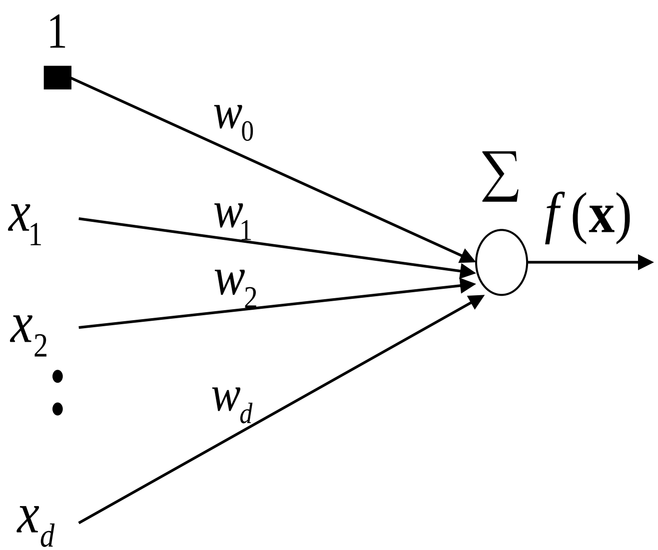
$\boldsymbol{\theta}$ - A location parameter $\boldsymbol{\varphi}$ - A scale parameter

- **Claim:** A logistic regression is a correct model when class conditional densities are from the same distribution in the exponential family and have **the same scale factor** $\boldsymbol{\varphi}$
- **Very powerful result !!!!**
  - **We can represent posteriors of many distributions with the same small network**

Linear units

Linear regression

$$f(\mathbf{x}) = \mathbf{w}^T \mathbf{x}$$



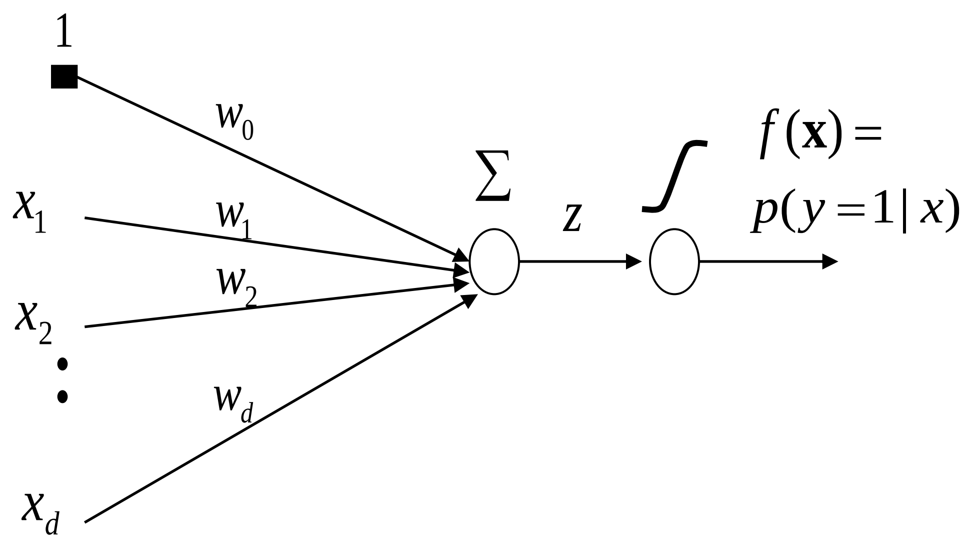
Gradient update:

$$\mathbf{w} \leftarrow \mathbf{w} + \alpha \sum_{i=1}^{n} (y_i - f(\mathbf{x}_i))\mathbf{x}_i$$

Online: $\mathbf{w} \leftarrow \mathbf{w} + \alpha (y - f(\mathbf{x}))\mathbf{x}$

Logistic regression

$$f(\mathbf{x}) = p(y = 1 | \mathbf{x}, \mathbf{w}) = g(\mathbf{w}^T \mathbf{x})$$



The same

Gradient update:

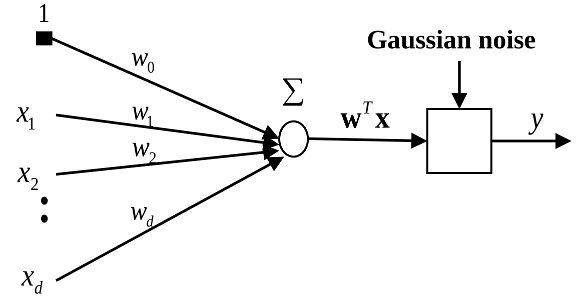
$$\mathbf{w} \leftarrow \mathbf{w} + \alpha \sum_{i=1}^{n} (y_i - f(\mathbf{x}_i))\mathbf{x}_i$$

Online: $\mathbf{w} \leftarrow \mathbf{w} + \alpha (y - f(\mathbf{x}))\mathbf{x}$

Gradient-based learning

- The **same simple gradient update rule** derived for both the linear and logistic regression models
- Where the magic comes from?
- Under the **log-likelihood** measure the function models and the models for the output selection fit together:
  - **Linear model + Gaussian noise**

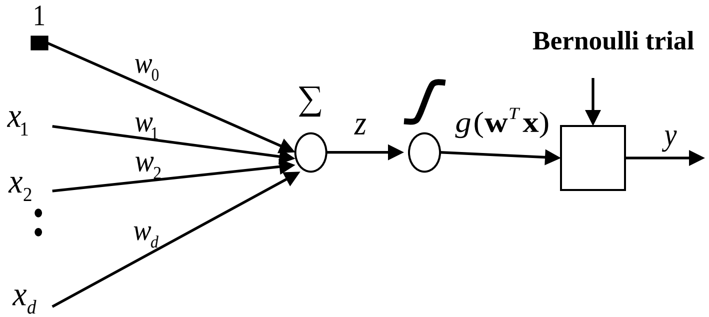
$$y = \mathbf{w}^T \mathbf{x} + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2)$$



  - **Logistic + Bernoulli**

$$y = \text{Bernoulli}(\theta)$$

$$\theta = p(y = 1 \mid \mathbf{x}) = g(\mathbf{w}^T \mathbf{x})$$



Generalized linear models (GLIM)

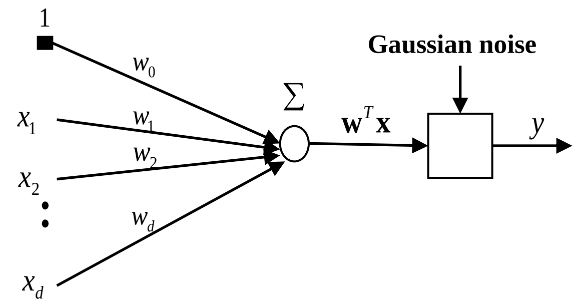
Assumptions:

- The conditional mean (expectation) is:

$$\mu = f(\mathbf{w}^T \mathbf{x})$$

  – Where $f(.)$ is a **response function**

- Output y is characterized by an exponential family distribution with a conditional mean μ



Examples:

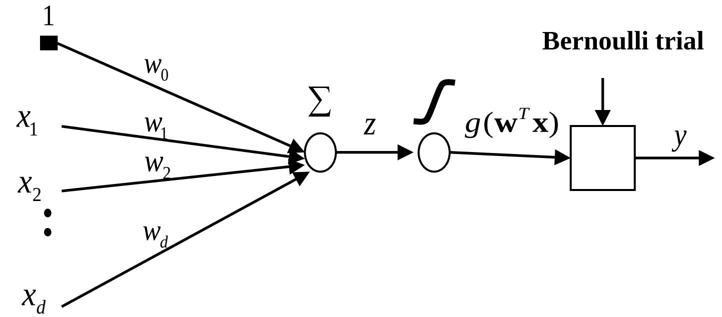
– **Linear model + Gaussian noise**

$$y = \mathbf{w}^T \mathbf{x} + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2)$$

– **Logistic + Bernoulli**

$$y \approx \text{Bernoulli}(\theta)$$

$$\theta = g(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + e^{-\mathbf{w}^T \mathbf{x}}}$$



Generalized linear models

- **A canonical response functions $f(.)$:**
  - **encoded in the distribution**

$$p(\mathbf{x} \mid \boldsymbol{\theta}, \boldsymbol{\varphi}) = h(x, \boldsymbol{\varphi}) \exp\left\{ \frac{\boldsymbol{\theta}^T \mathbf{x} - A(\boldsymbol{\theta})}{a(\boldsymbol{\varphi})} \right\}$$

- **Leads to a simple gradient form**
- **Example: Bernoulli distribution**

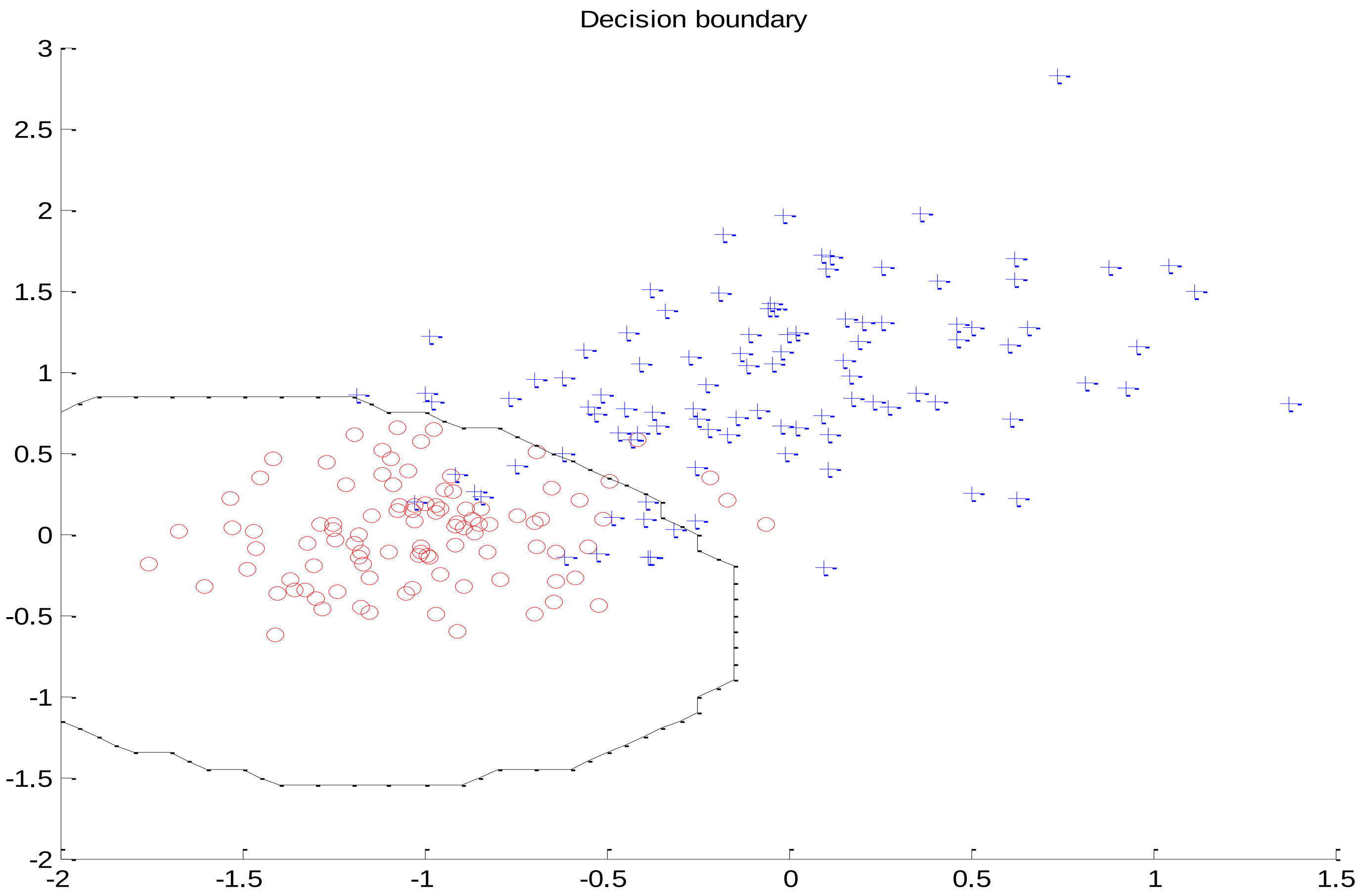
$$p(x \mid \mu) = \mu^x (1-\mu)^{1-x} = \exp\left\{ \log\left(\frac{\mu}{1-\mu}\right) x + \log(1-\mu) \right\}$$

$$\theta = \log\left(\frac{\mu}{1-\mu}\right) \qquad \mu = \frac{1}{1+e^{-\theta}}$$

  - **Logistic function matches the Bernoulli**

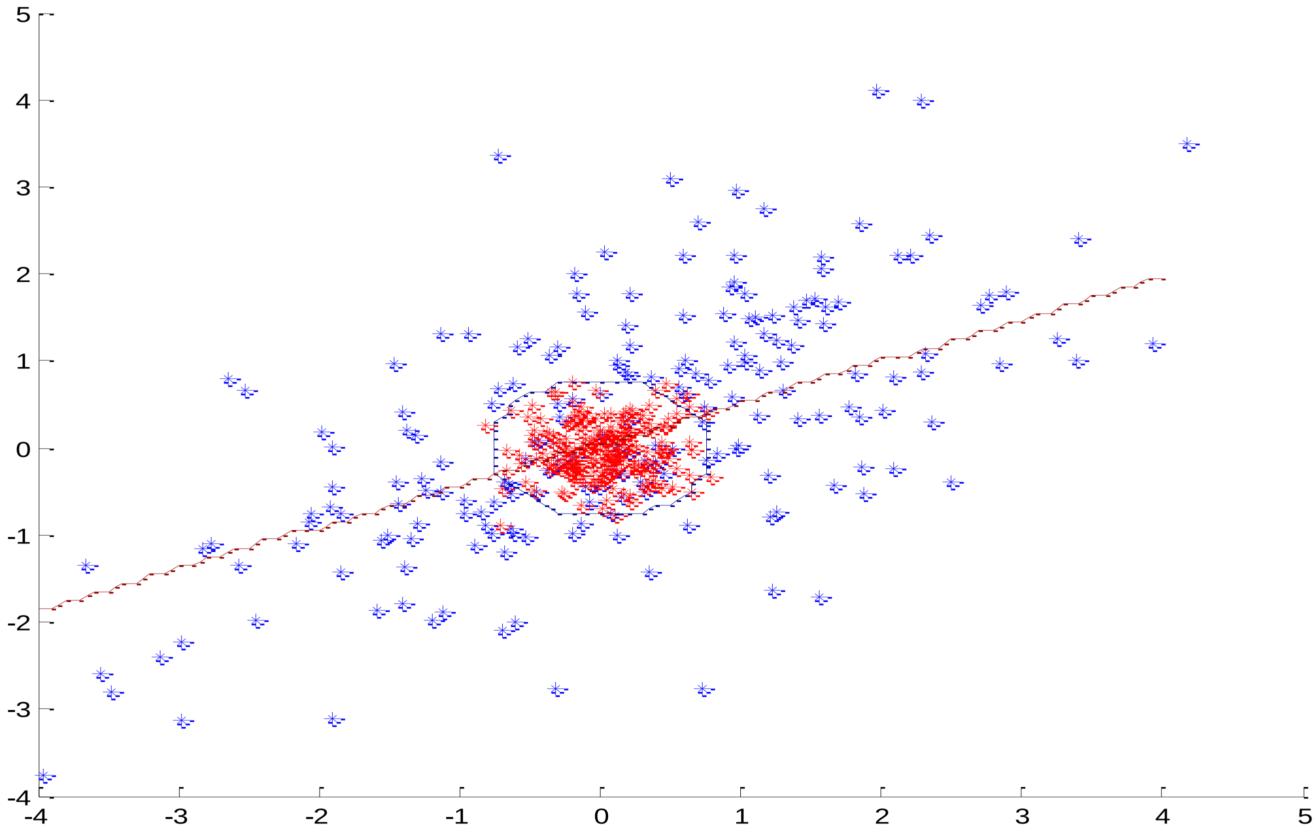
When does the logistic regression fail?

- Quadratic decision boundary is needed



When does the logistic regression fail?

- Another example of a non-linear decision boundary



Non-linear extension of logistic regression

- use **feature (basis) functions** to model **nonlinearities**
  - the same trick as used for the linear regression

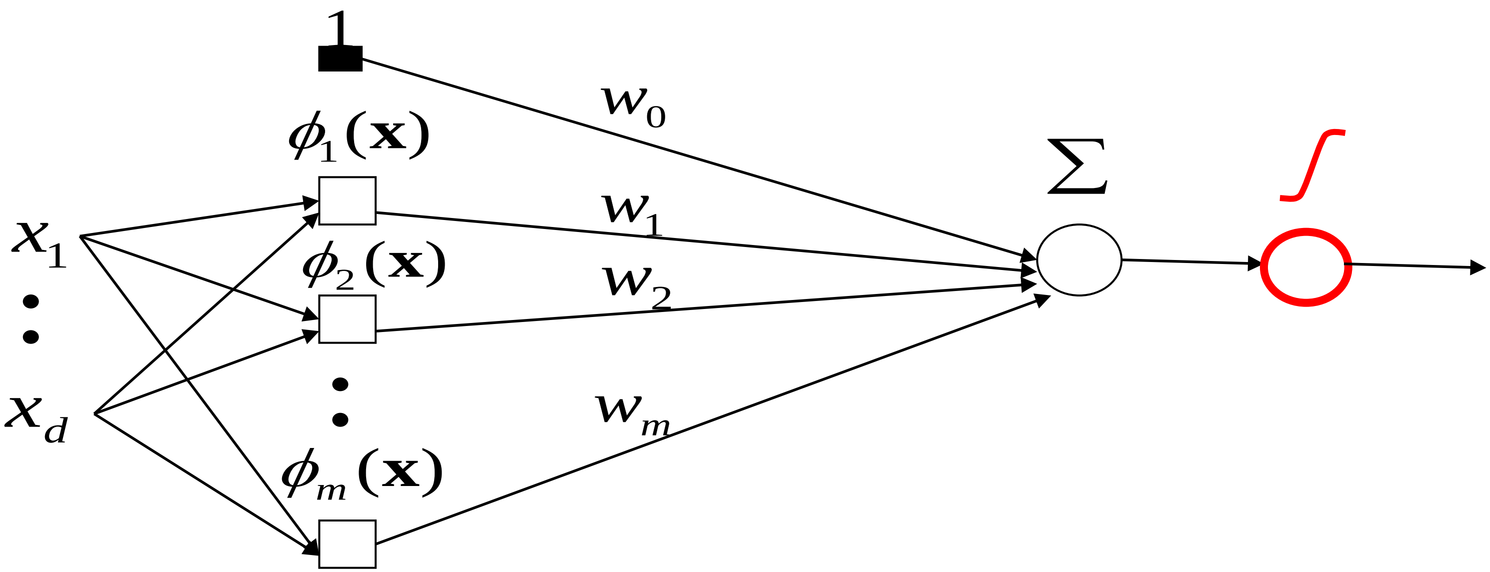
Linear regression

$$f(\mathbf{x}) = w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x})$$

Logistic regression

$$f(\mathbf{x}) = g(w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x}))$$

$\phi_j(\mathbf{x})$ - an arbitrary function of $\mathbf{x}$



Evaluation of classifiers

Evaluation

For any data set we use to test the classification model on we can build a **confusion matrix:**

- Counts of examples with:
- class label ω_j that are classified with a label α_i

target

		$\omega = 1$	$\omega = 0$
predict	$\alpha = 1$	140	17
	$\alpha = 0$	20	54

Evaluation

For any data set we use to test the model we can build a **confusion matrix:**

		target	
		$\omega = 1$	$\omega = 0$
predict	$\alpha = 1$	140	17
	$\alpha = 0$	20	54

agreement

Error: ?

Evaluation

For any data set we use to test the model we can build a confusion matrix:

target

predict

	$\omega=1$	$\omega=0$
$\alpha=1$	140	17
$\alpha=0$	20	54

agreement

Error: = 37/231

Accuracy = 1- Error = 194/231

Evaluation for binary classification

Entries in the confusion matrix for binary classification have names:

target

predict		$\omega = 1$	$\omega = 0$
	$\alpha = 1$	TP	FP
	$\alpha = 0$	FN	TN

TP: True positive (hit)

FP: False positive (false alarm)

TN: True negative (correct rejection)

FN: False negative (a miss)

Additional statistics

- **Sensitivity (recall)**

$$SENS = \frac{TP}{TP + FN}$$

- **Specificity**

$$SPEC = \frac{TN}{TN + FP}$$

- **Positive predictive value (precision)**

$$PPT = \frac{TP}{TP + FP}$$

- **Negative predictive value**

$$NPV = \frac{TN}{TN + FN}$$

Binary classification: additional statistics

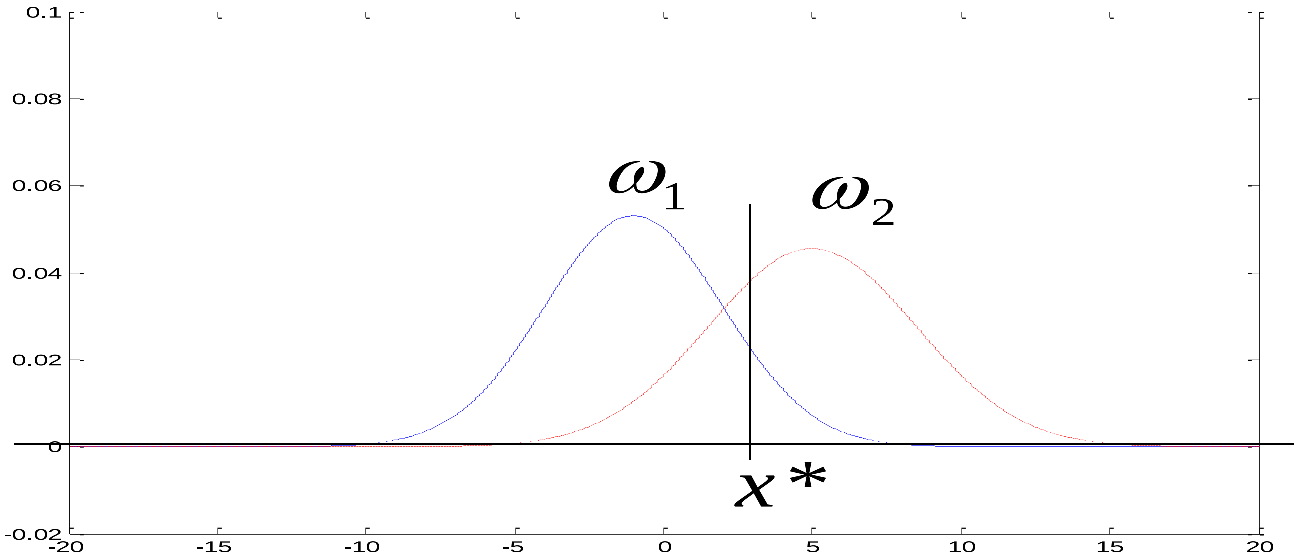
- **Confusion matrix**

		target		
		1	0	
predict	1	140	10	$PPV = 140/150$
	0	20	180	$NPV = 180/200$
		$SENS = 140/160$	$SPEC = 180/190$	

Row and column quantities:

- Sensitivity (SENS)
- Specificity (SPEC)
- Positive predictive value (PPV)
- Negative predictive value (NPV)

Binary decisions: Receiver Operating Curves



- **Probabilities:**
  - *SENS* $\quad p(x > x^* \mid \mathbf{x} \in \omega_2)$
  - *SPEC* $\quad p(x < x^* \mid \mathbf{x} \in \omega_1)$

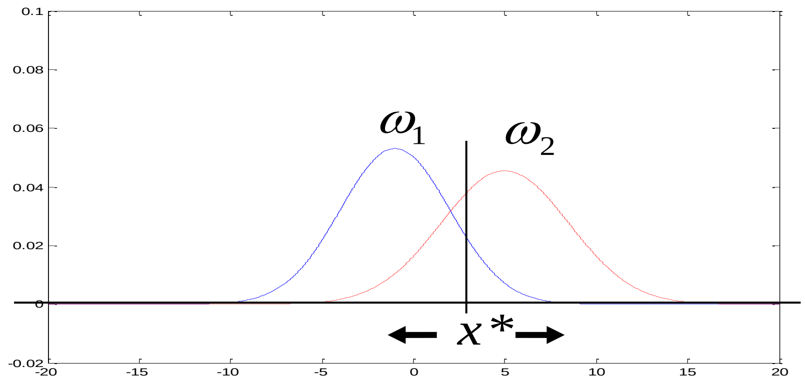
Receiver Operating Characteristic (ROC)

- **ROC curve plots :**

SN= $p(x > x^* \mid \mathbf{x} \in \omega_2)$

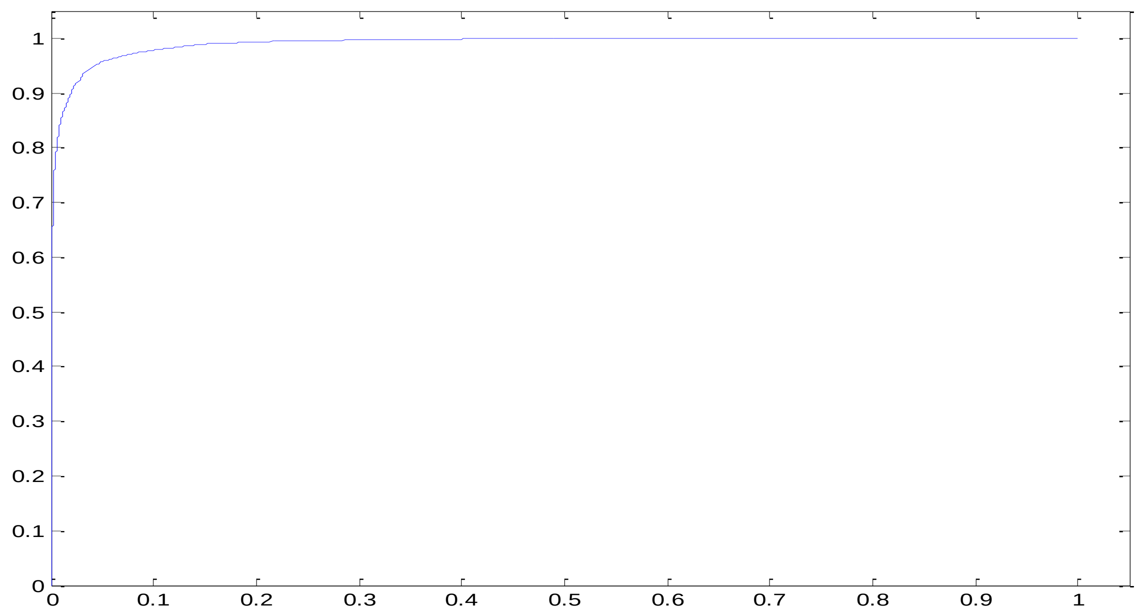
1-SP= $p(x > x^* \mid \mathbf{x} \in \omega_1)$

for different x*



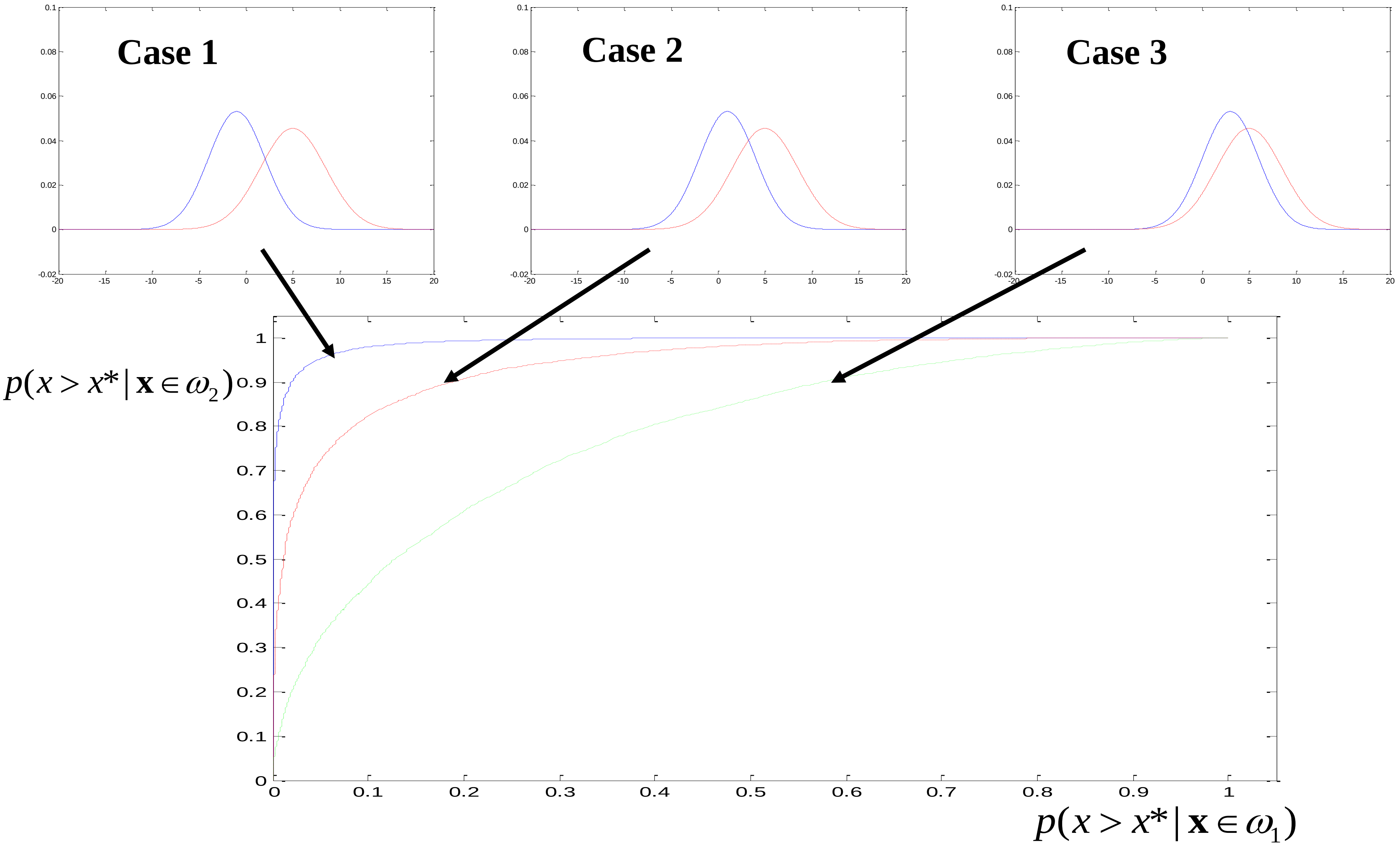
SENS

$p(x > x^* \mid \mathbf{x} \in \omega_2)$



1-SPEC $p(x > x^* \mid \mathbf{x} \in \omega_1)$

ROC curve



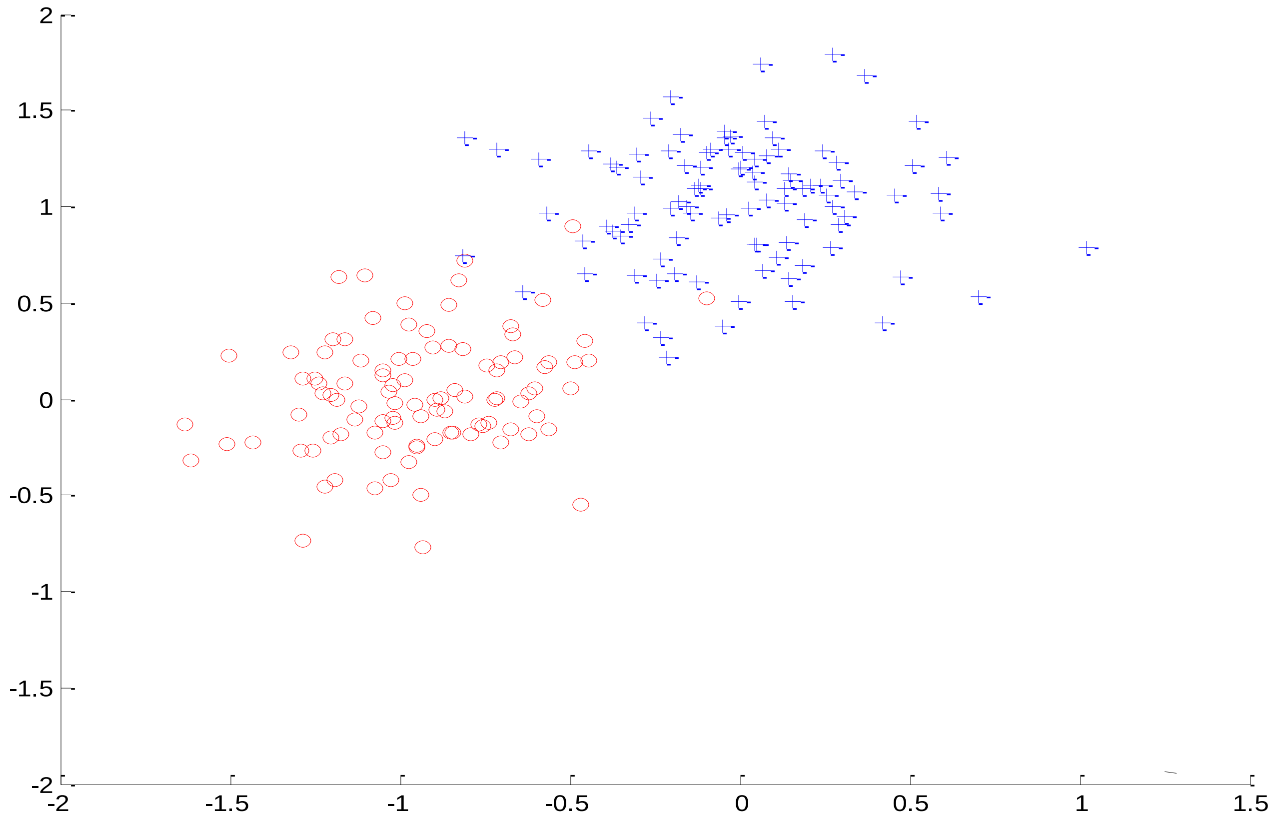
Case 1
Case 2
Case 3
$p(x > x^* \mid \mathbf{x} \in \omega_2)$
$p(x > x^* \mid \mathbf{x} \in \omega_1)$

Receiver operating characteristic

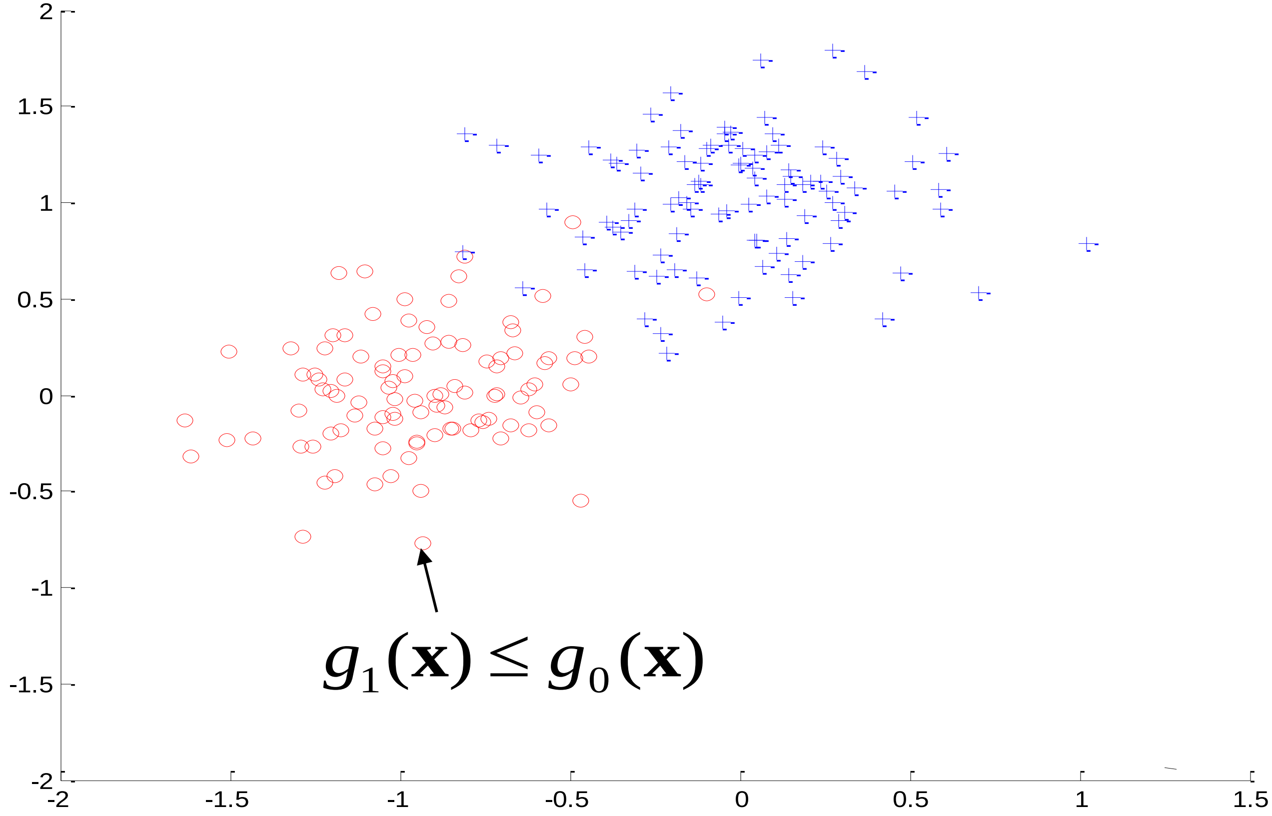
- **ROC**
  – shows the discriminability between the two classes under different decision biases
- **Decision bias**
  – can be changed using different loss function

Back to classification models

Discriminant functions

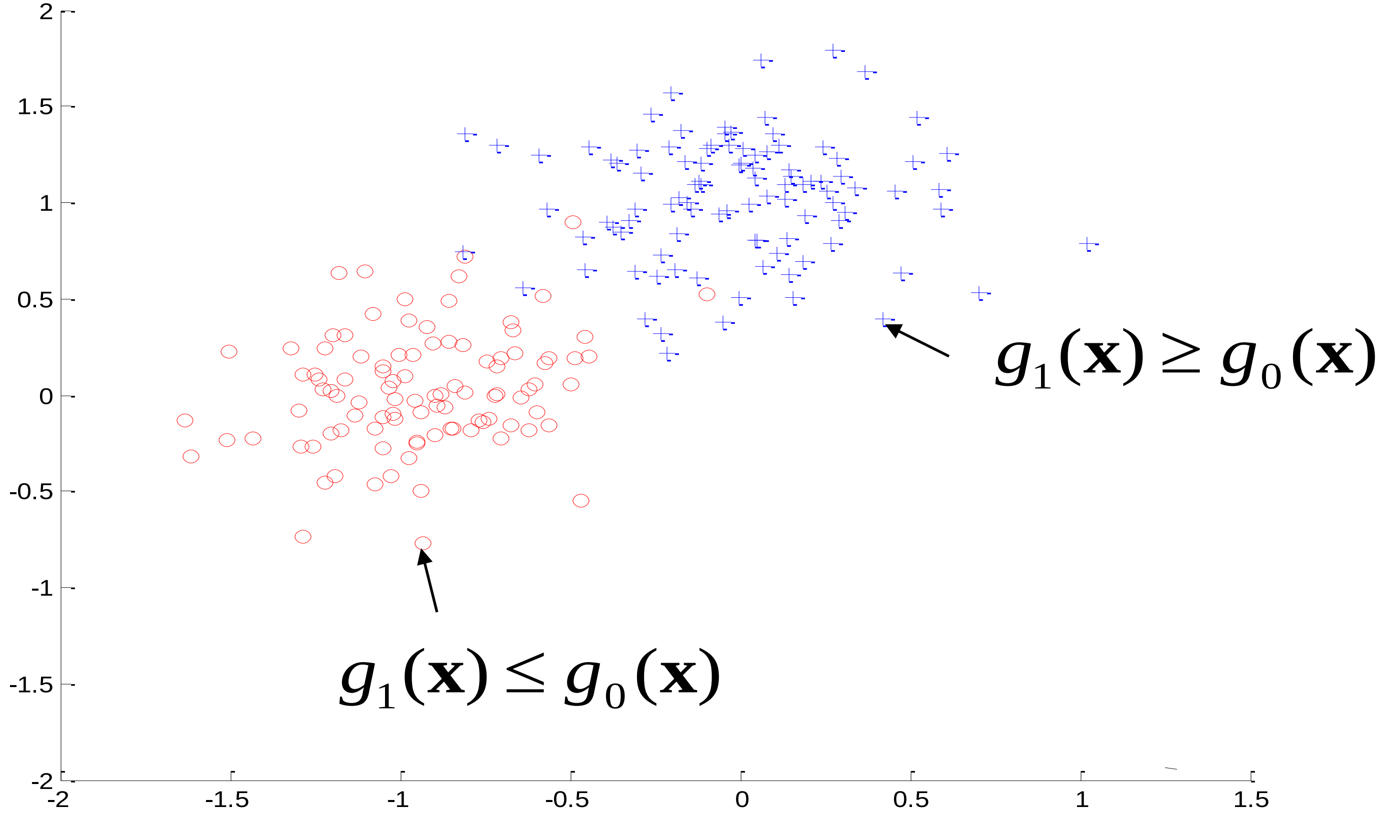


Discriminant functions



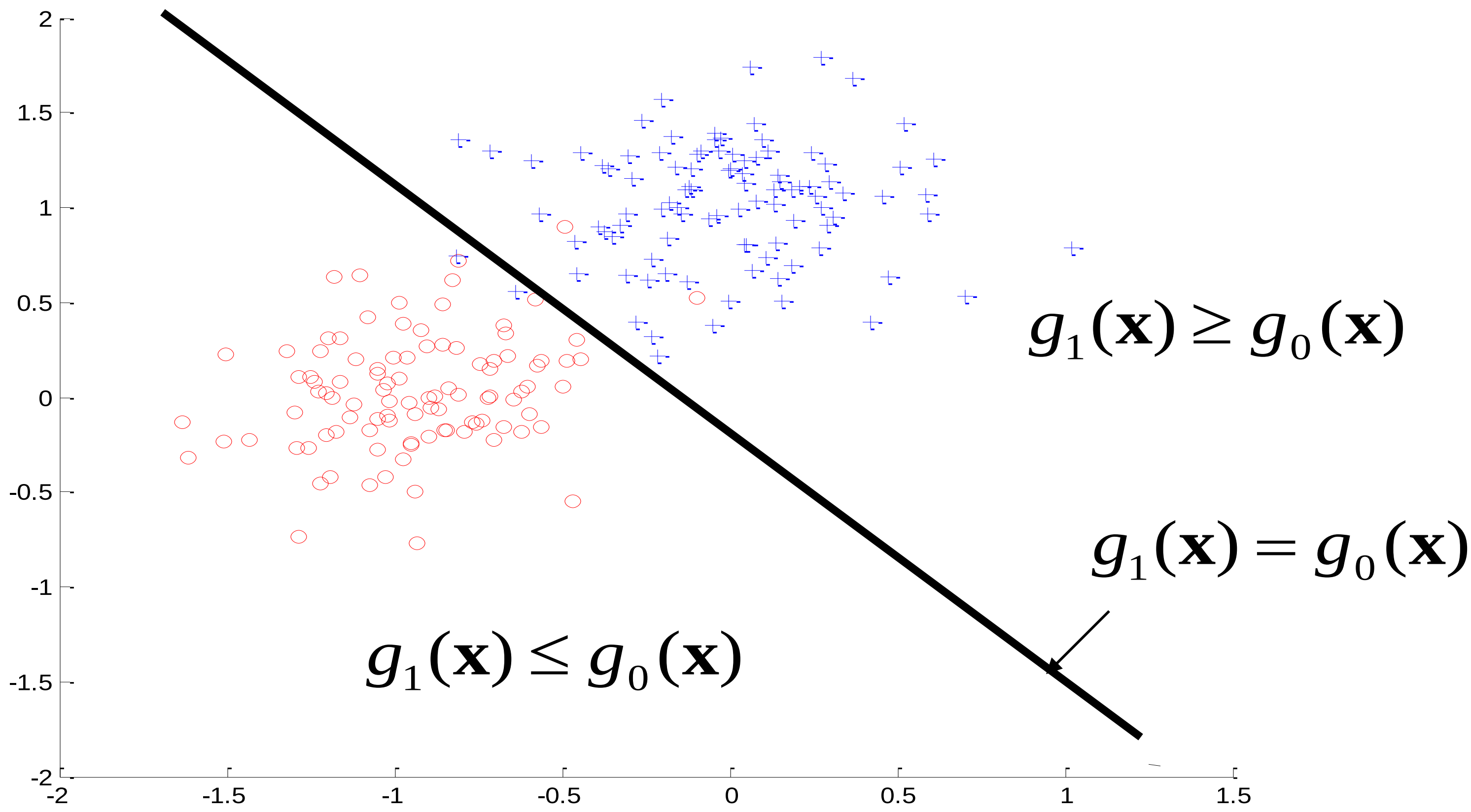
$$g_1(\mathbf{x}) \le g_0(\mathbf{x})$$

Discriminant functions

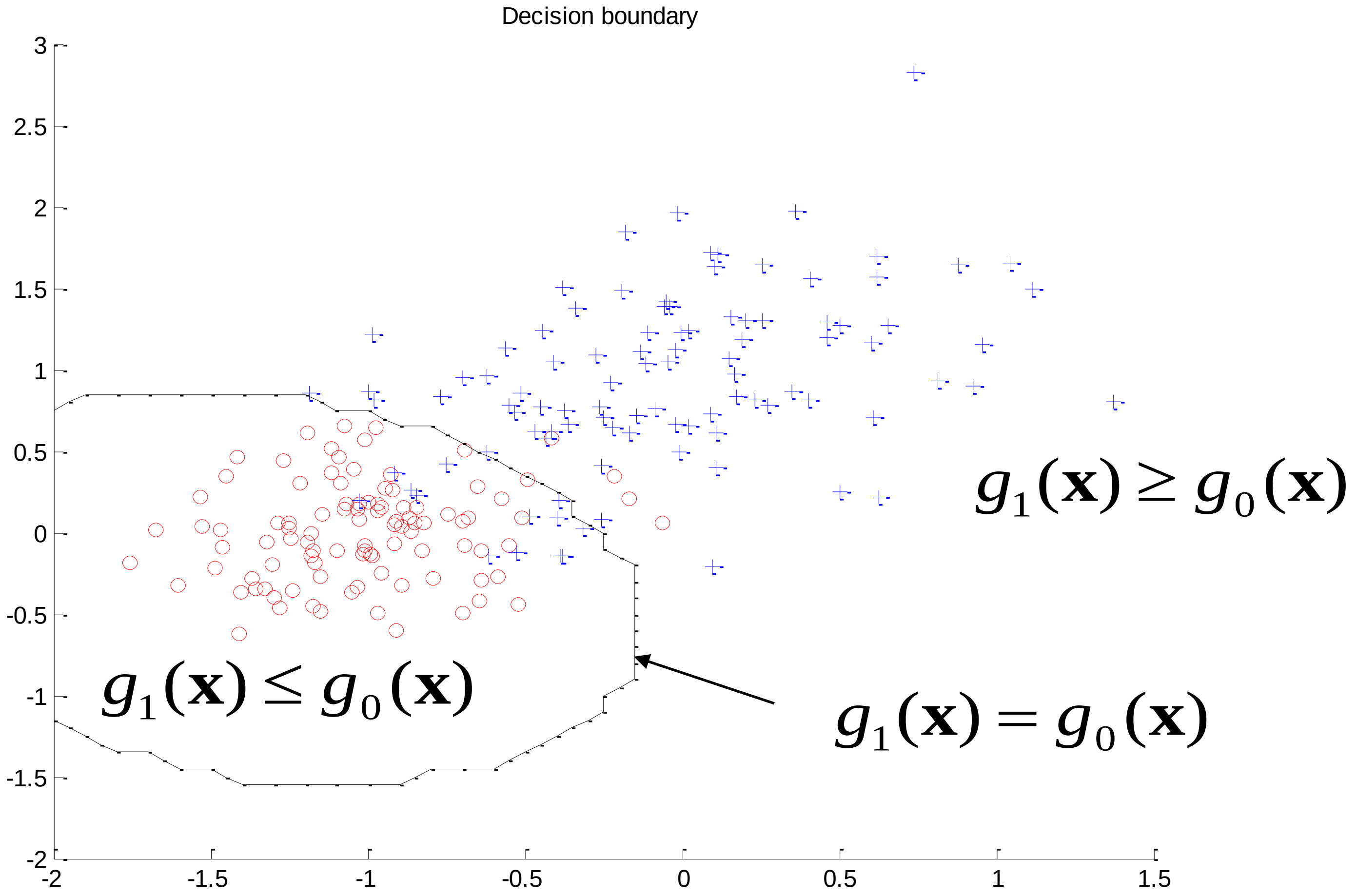


Discriminant functions

- Define **decision boundary**



Quadratic decision boundary



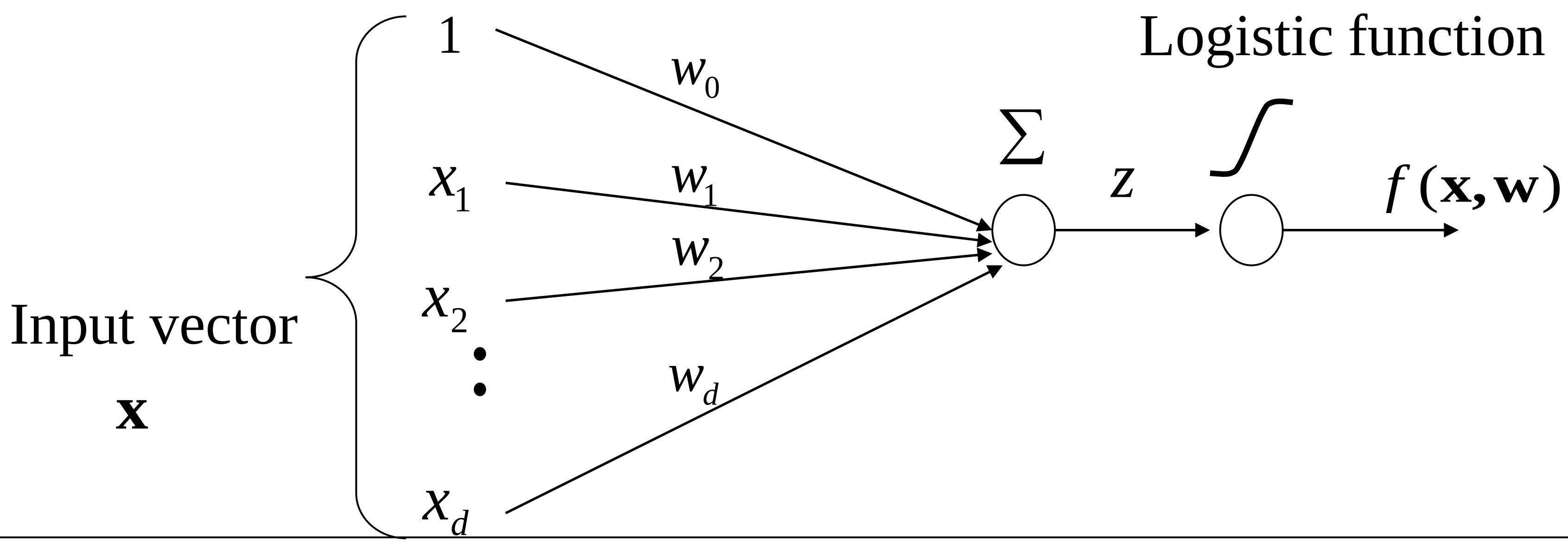
Logistic regression model

- **Defines a linear decision boundary**
- **Discriminant functions:**

$$g_1(\mathbf{x}) = g(\mathbf{w}^T\mathbf{x}) \qquad\qquad g_0(\mathbf{x}) = 1 - g(\mathbf{w}^T\mathbf{x})$$

- **where** $g(z) = 1/(1+e^{-z})$ - is a logistic function

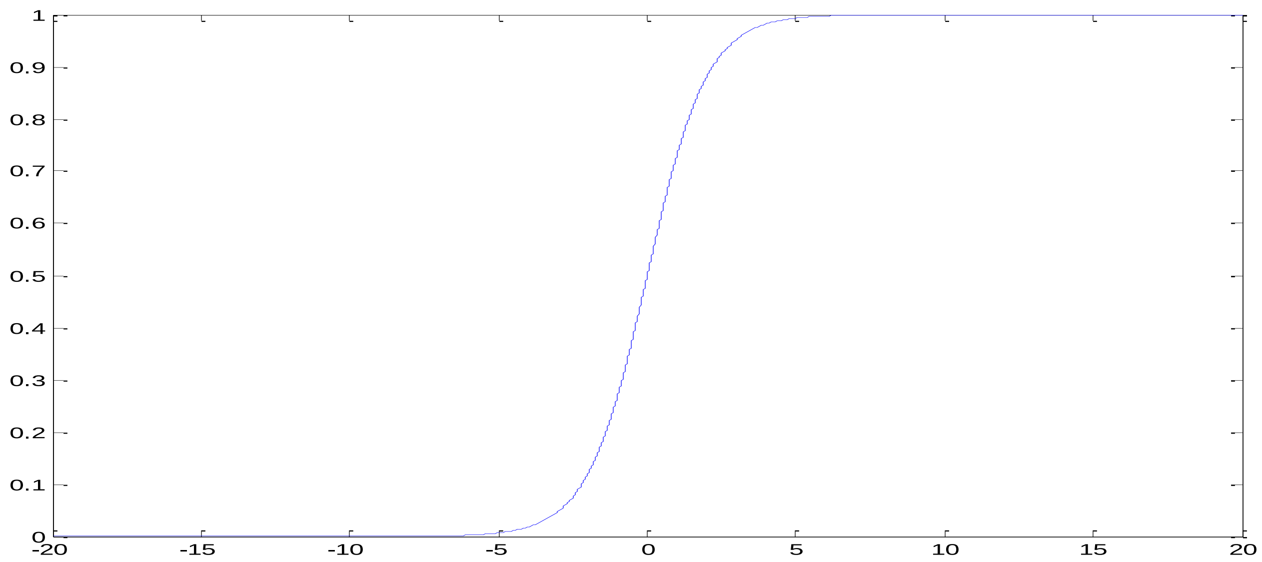
$$f(\mathbf{x},\mathbf{w}) = g_1(\mathbf{w}^T\mathbf{x}) = g(\mathbf{w}^T\mathbf{x})$$



Logistic function

function $$g(z) = \frac{1}{(1+e^{-z})}$$

- Is also referred to as a **sigmoid function**
- Replaces the threshold function with smooth switching
- takes a real number and outputs the number in the interval [0,1]



Generative approach to classification

Idea:

1. **Represent and learn the distribution** $p(\mathbf{x}, y)$
2. **Use it to define probabilistic discriminant functions**

E.g. $g_o(\mathbf{x}) = p(y=0 \mid \mathbf{x}) \qquad g_1(\mathbf{x}) = p(y=1 \mid \mathbf{x})$

Typical model $\qquad p(\mathbf{x}, y) = p(\mathbf{x} \mid y)\, p(y)$

- $p(\mathbf{x} \mid y)$ **= Class-conditional distributions (densities)**

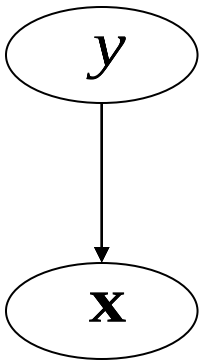
  binary classification: two class-conditional distributions

  $$p(\mathbf{x} \mid y=0) \qquad\qquad p(\mathbf{x} \mid y=1)$$

- $p(y)$ **= Priors on classes -** probability of class y

  binary classification: Bernoulli distribution

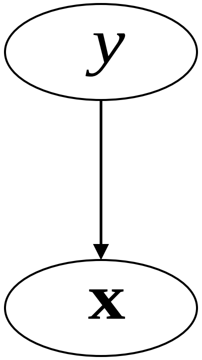
  $$p(y=0) + p(y=1) = 1$$



Quadratic discriminant analysis (QDA)

Model:

- **Class-conditional distributions**
  - **multivariate normal distributions**



$$\mathbf{x} \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \quad \text{for} \quad y = 0$$

$$\mathbf{x} \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) \quad \text{for} \quad y = 1$$

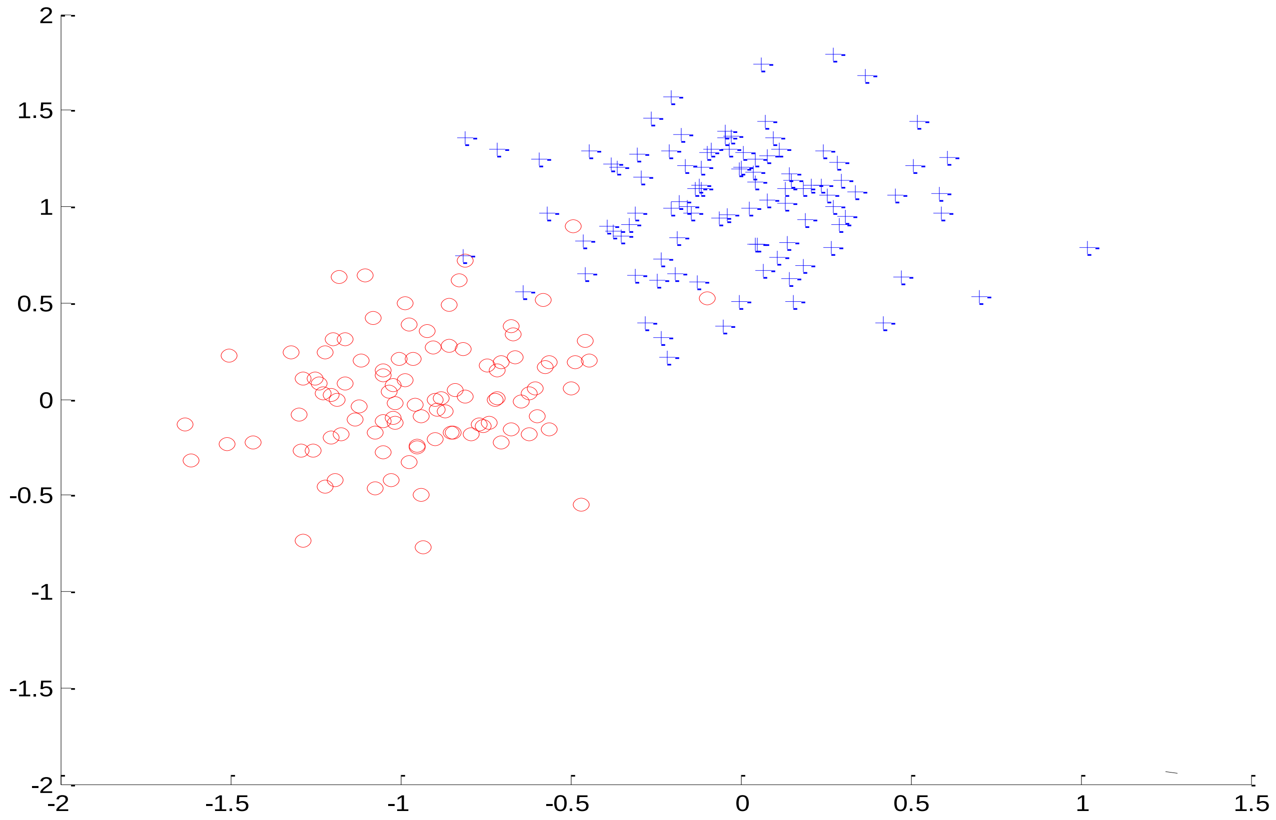
Multivariate normal $\quad \mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]$$

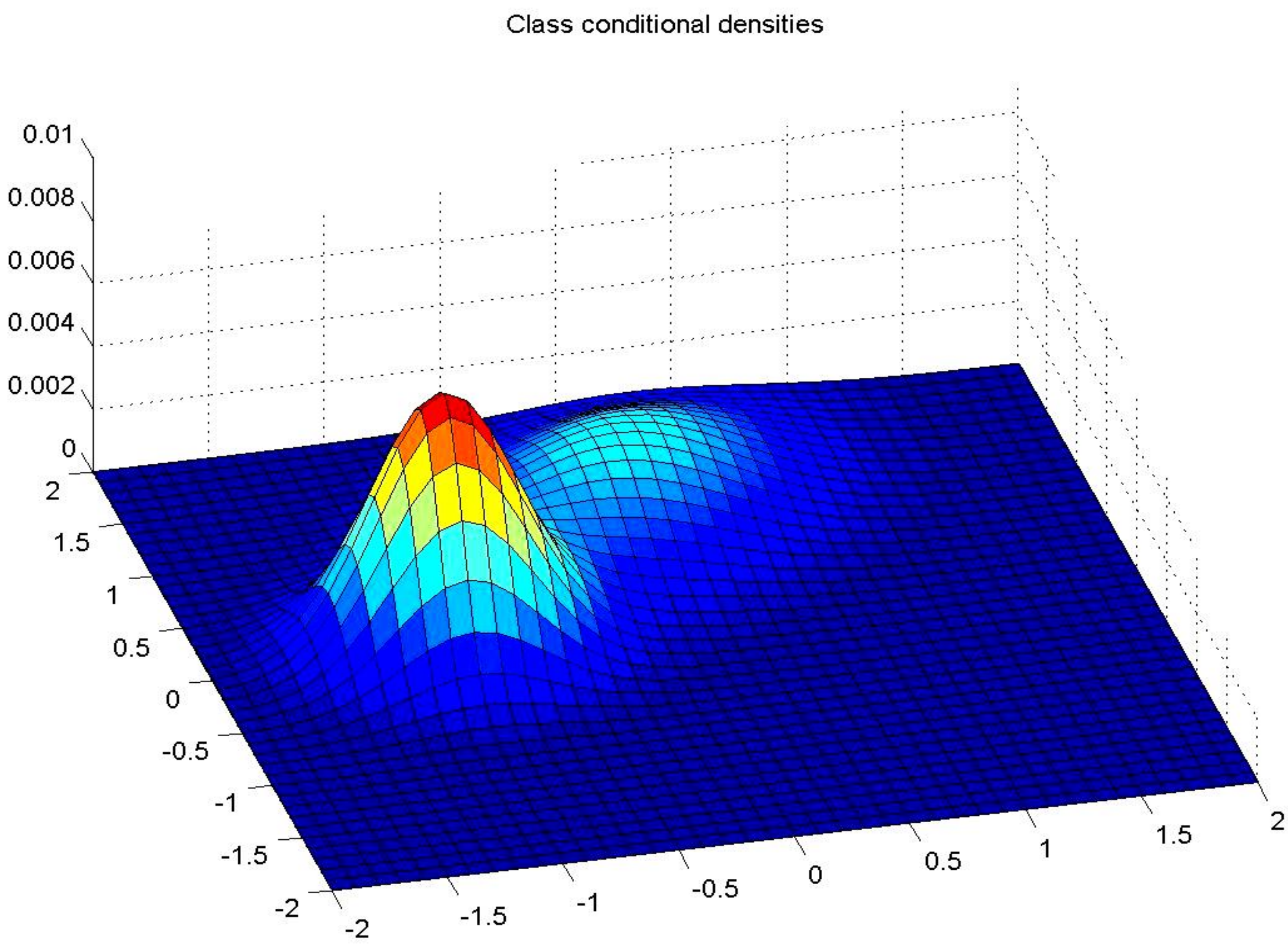
- **Priors on classes (class 0,1)** $\quad y \sim Bernoulli$
  - **Bernoulli distribution**

$$p(y, \theta) = \theta^y (1 - \theta)^{1-y} \qquad y \in \{0,1\}$$

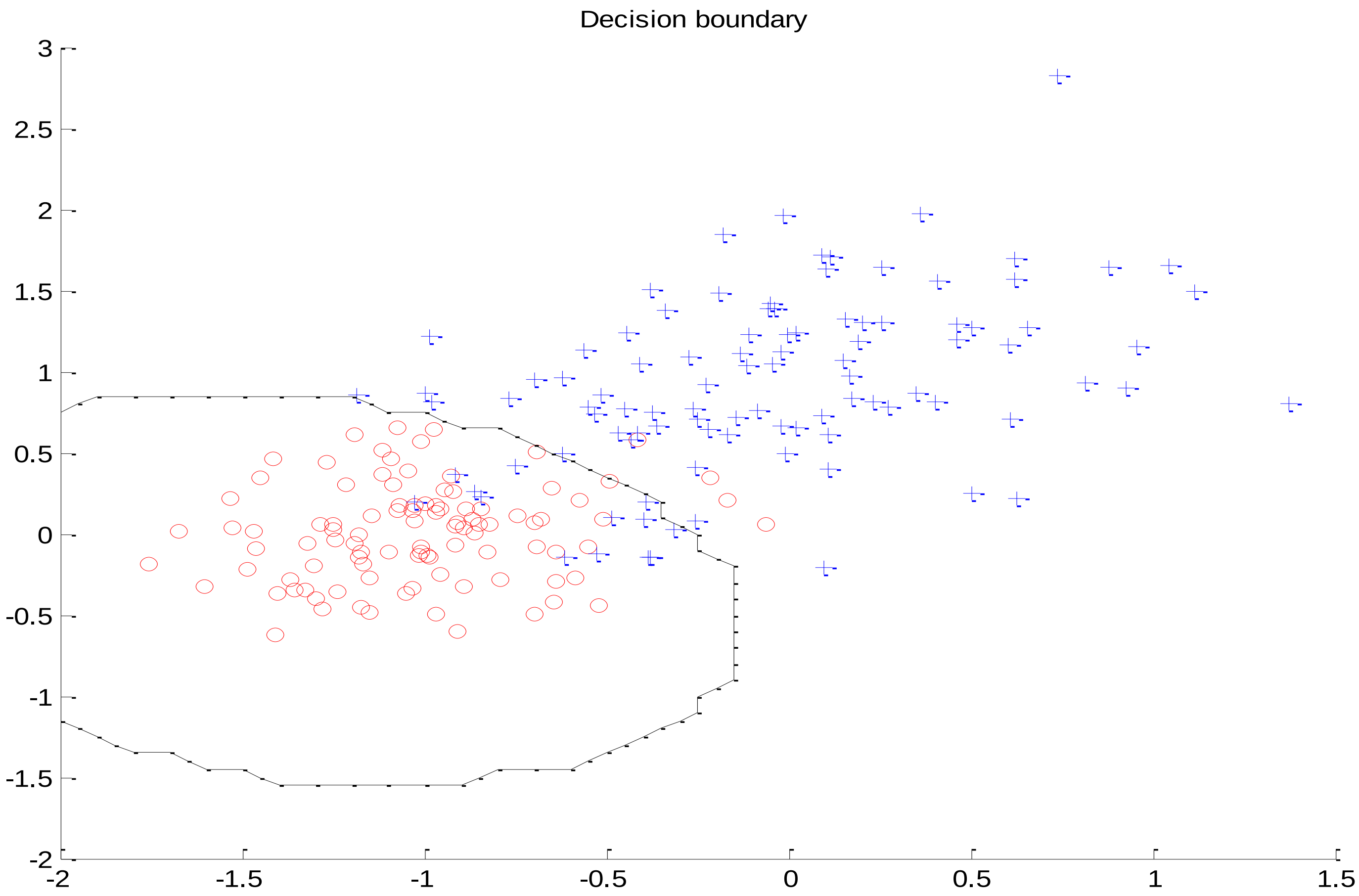
QDA



2 Gaussian class-conditional densities



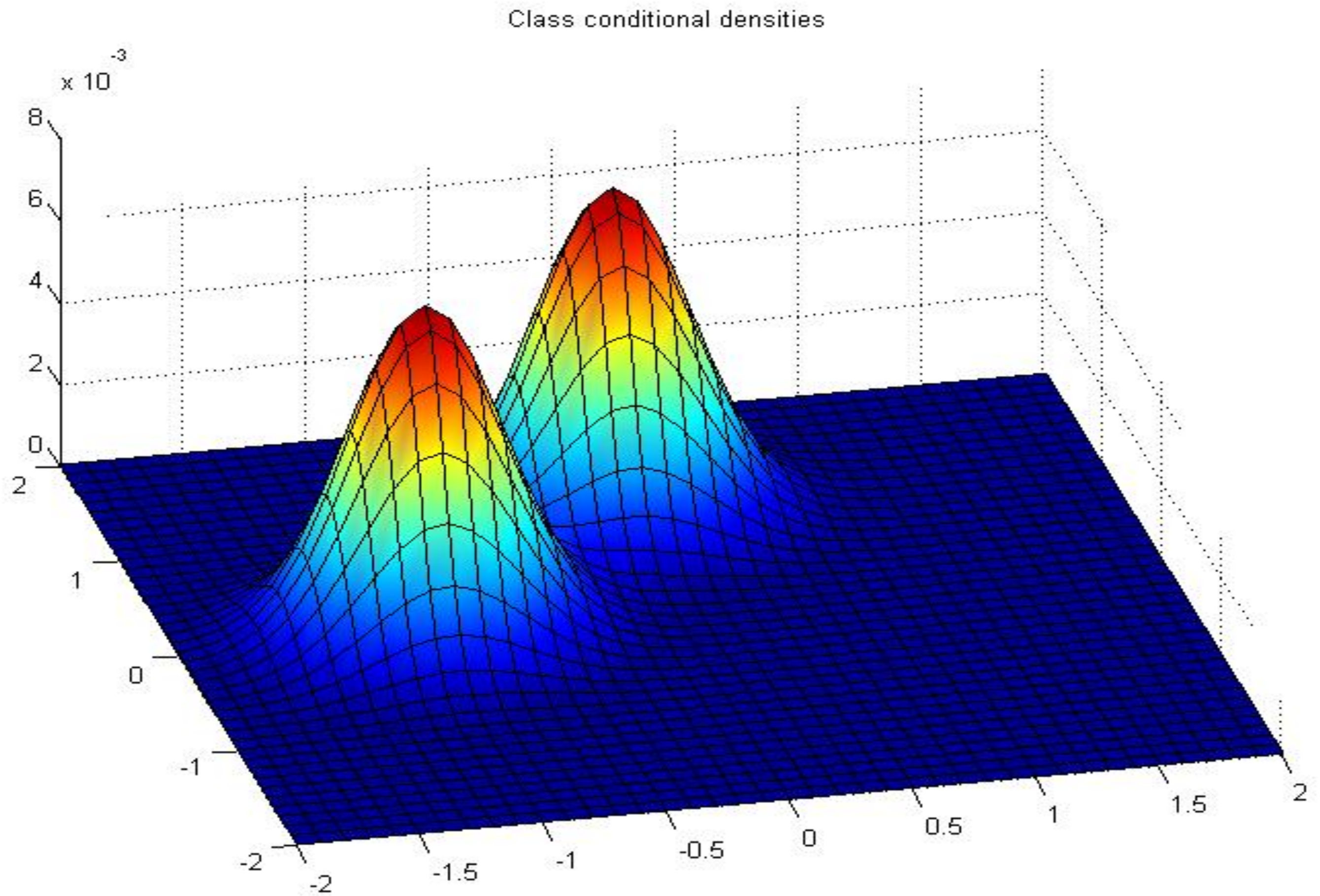
QDA: Quadratic decision boundary



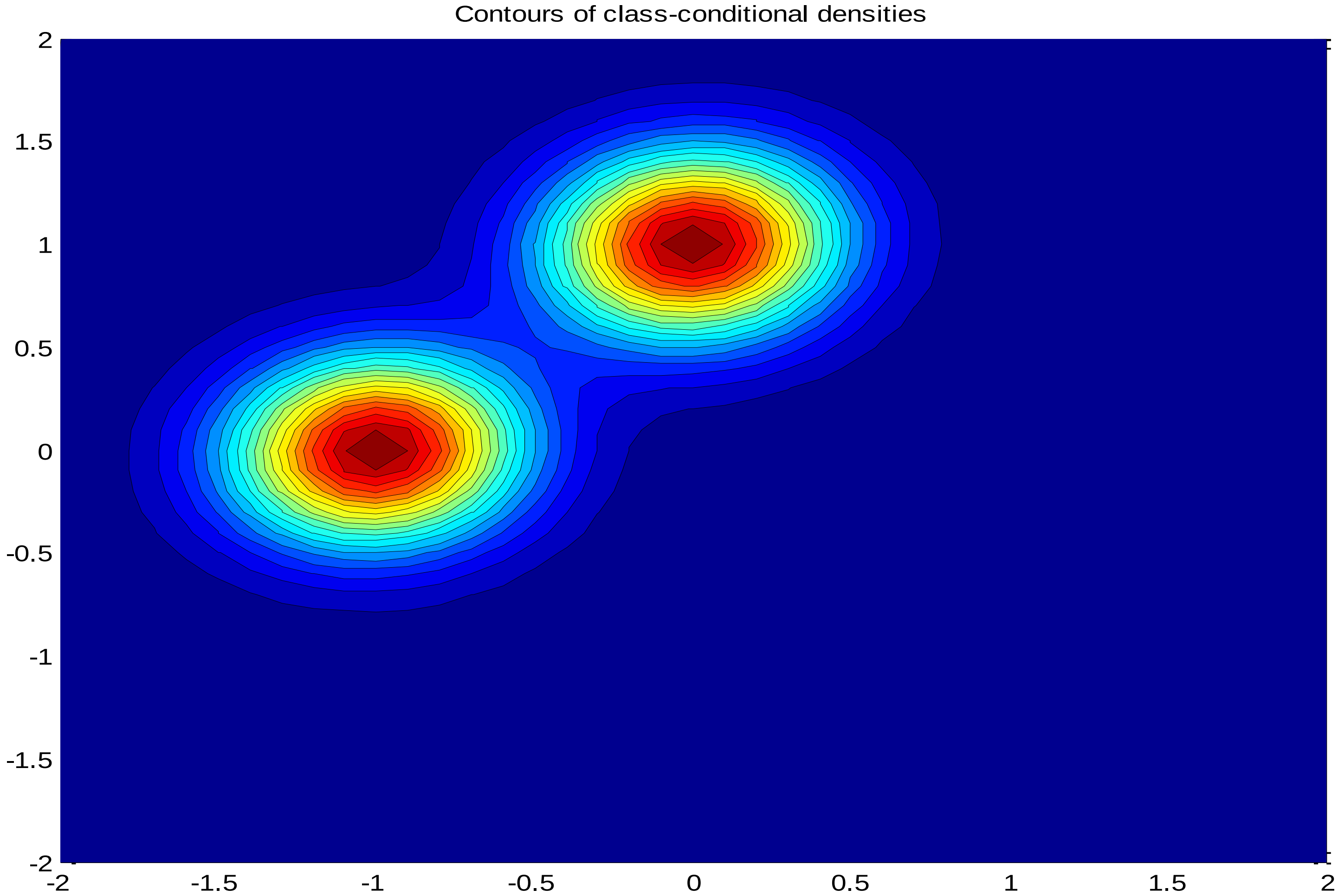
Linear discriminant analysis (LDA)

- When covariances are the same

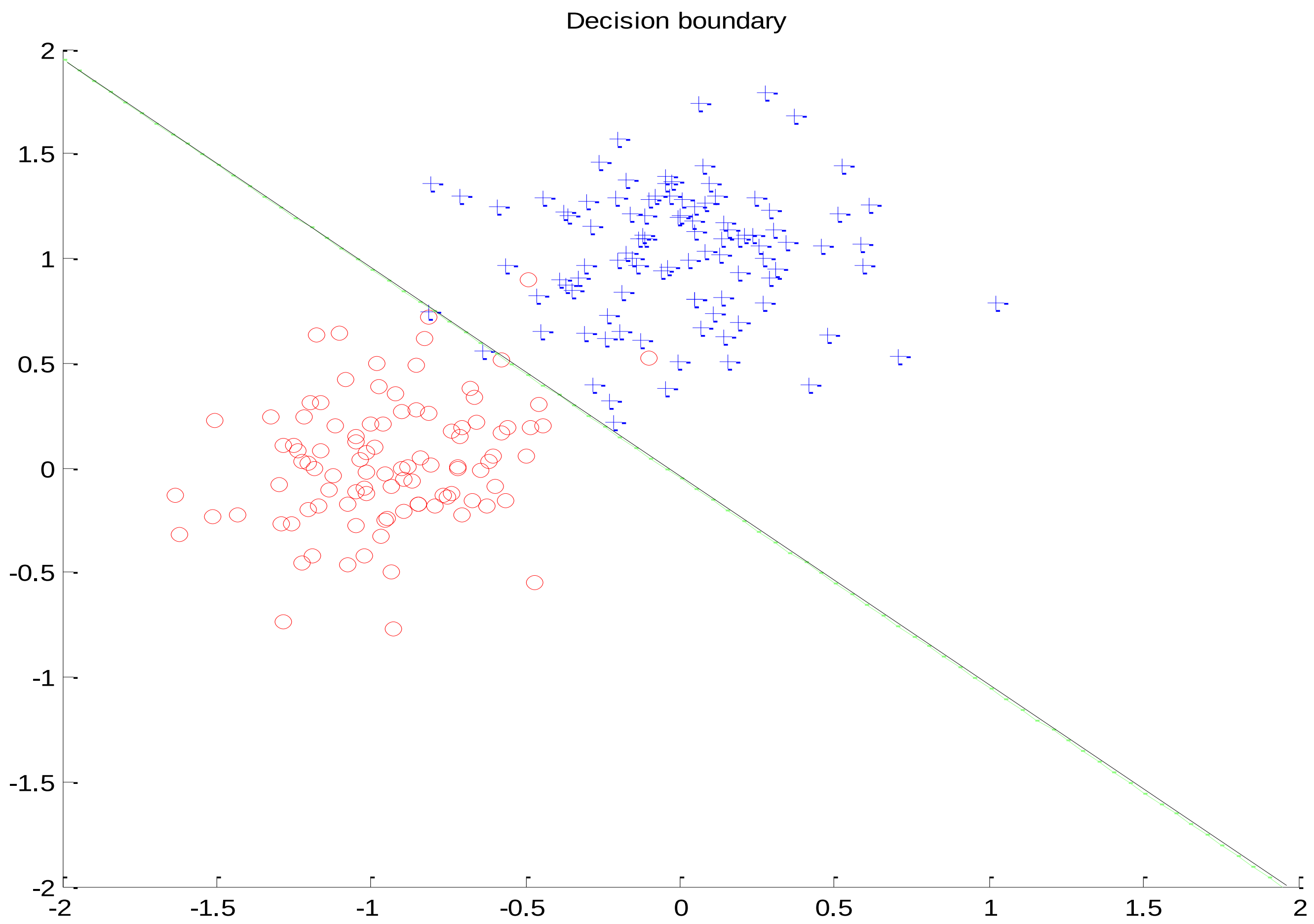
$$\mathbf{x} \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}),\ y = 0$$
$$\mathbf{x} \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}),\ y = 1$$



LDA: Linear decision boundary



LDA: linear decision boundary



Logistic regression vs LDA

- **Two models with linear decision boundaries:**
  - **Logistic regression**
  - **Generative model with 2 Gaussians with the same covariance matrices**

$$x \sim N(\mu_0, \Sigma) \quad \text{for} \quad y = 0$$
$$x \sim N(\mu_1, \Sigma) \quad \text{for} \quad y = 1$$

- **Two models are related !!!**
  - When we have **2 Gaussians with the same covariance matrix** the probability of y given **x** has the form of a logistic regression model **!!!**

$$\boxed{p(y=1 \mid \mathbf{x}, \boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \boldsymbol{\Sigma}) = g(\mathbf{w}^T \mathbf{x})}$$

When is the logistic regression model correct?

- **Members of the exponential family can be often more naturally described as**

$$f(\mathbf{x} \mid \boldsymbol{\theta}, \boldsymbol{\varphi}) = h(x, \boldsymbol{\varphi}) \exp\left\{ \frac{\boldsymbol{\theta}^T \mathbf{x} - A(\boldsymbol{\theta})}{a(\boldsymbol{\varphi})} \right\}$$

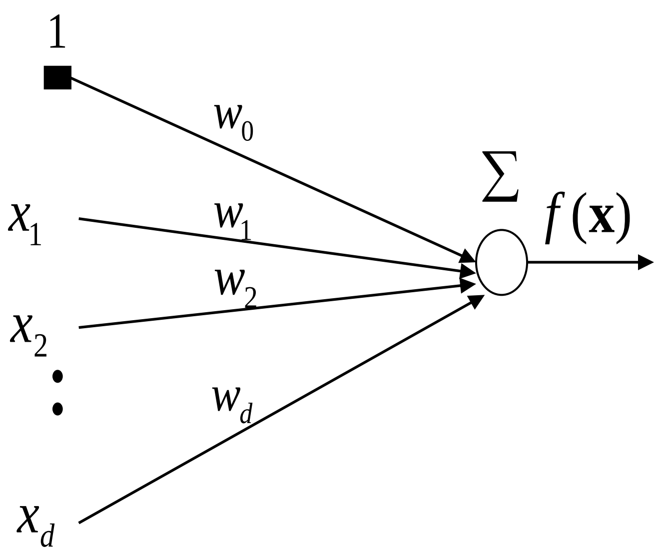
$\boldsymbol{\theta}$ - A location parameter $\boldsymbol{\varphi}$ - A scale parameter

- **Claim:** A logistic regression is a correct model when class conditional densities are from the same distribution in the exponential family and have **the same scale factor** $\boldsymbol{\varphi}$
- **Very powerful result !!!!**
  - **We can represent posteriors of many distributions with the same small network**

Linear units

Linear regression

$$f(\mathbf{x}) = \mathbf{w}^T \mathbf{x}$$



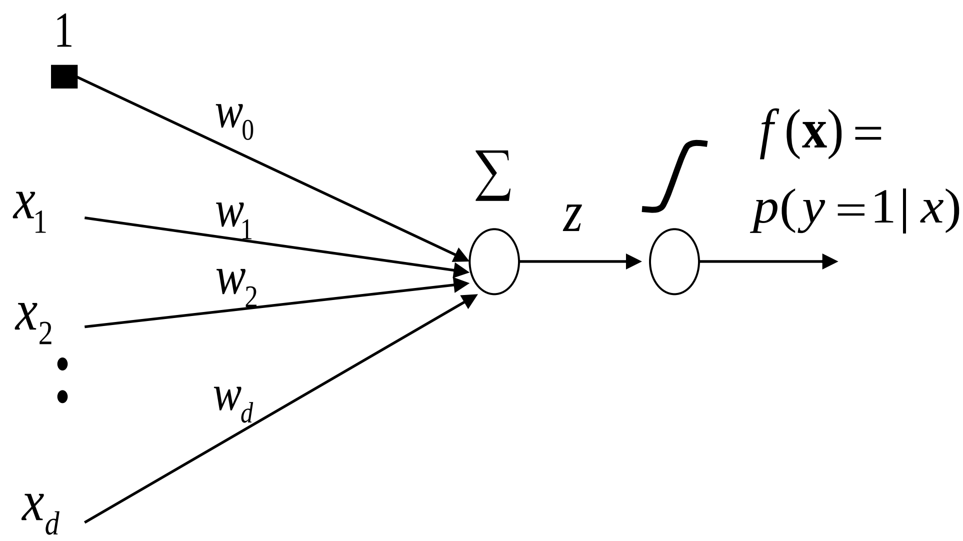
Gradient update:

$$\mathbf{w} \leftarrow \mathbf{w} + \alpha \sum_{i=1}^{n} (y_i - f(\mathbf{x}_i))\mathbf{x}_i$$

Online: $\mathbf{w} \leftarrow \mathbf{w} + \alpha(y - f(\mathbf{x}))\mathbf{x}$

Logistic regression

$$f(\mathbf{x}) = p(y = 1 \mid \mathbf{x}, \mathbf{w}) = g(\mathbf{w}^T \mathbf{x})$$



Gradient update:

$$\mathbf{w} \leftarrow \mathbf{w} + \alpha \sum_{i=1}^{n} (y_i - f(\mathbf{x}_i))\mathbf{x}_i$$

Online: $\mathbf{w} \leftarrow \mathbf{w} + \alpha(y - f(\mathbf{x}))\mathbf{x}$

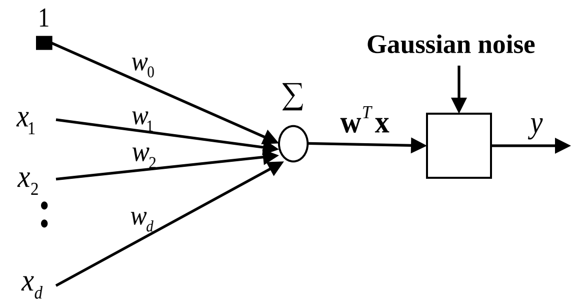
The same

Gradient-based learning

- The **same simple gradient update rule** derived for both the linear and logistic regression models
- Where the magic comes from?
- Under the **log-likelihood** measure the function models and the models for the output selection fit together:
  - **Linear model + Gaussian noise**

$$y = \mathbf{w}^T \mathbf{x} + \varepsilon$$

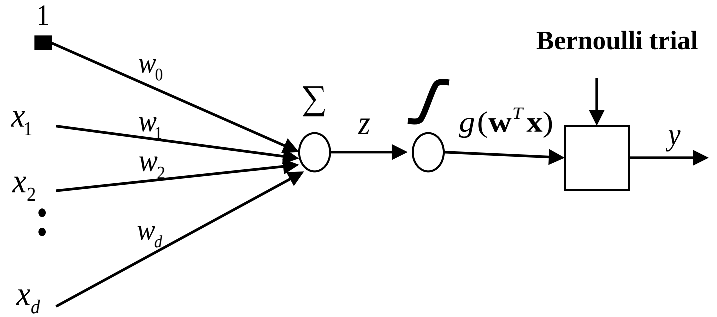
$$\varepsilon \sim N(0, \sigma^2)$$



  - **Logistic + Bernoulli**

$$y \sim \text{Bern}(\theta)$$

$$\theta = p(y = 1 \mid \mathbf{x}) = g(\mathbf{w}^T \mathbf{x})$$



Generalized linear models (GLIM)

Assumptions:

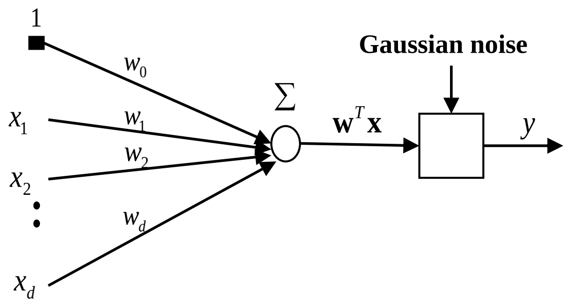
- The conditional mean (expectation) is: $\mu = f(\mathbf{w}^T \mathbf{x})$
  - $f(.)$ is a **response (or a link) function**
- Output y is characterized by an exponential family distribution with mean $\mu = f(\mathbf{w}^T \mathbf{x})$

Examples:

- **Linear model + Gaussian noise**

$$y = \mathbf{w}^T \mathbf{x} + \varepsilon \quad \varepsilon \sim N(0, \sigma^2)$$

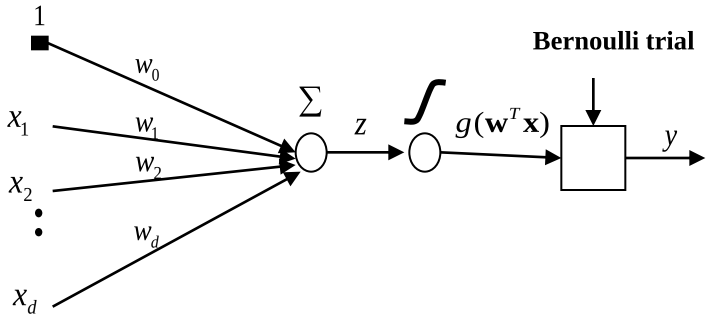
$$y \sim N(\mathbf{w}^T \mathbf{x}, \sigma^2)$$



- **Logistic + Bernoulli**

$$y \sim \mathrm{Bern}(\theta) \sim \mathrm{Bern}(g(\mathbf{w}^\mathrm{T} \mathbf{x}))$$

$$\theta = g(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + e^{-\mathbf{w}^T \mathbf{x}}}$$



Generalized linear models (GLMs)

- **A canonical response functions $f(.)$:**
  - **encoded in the distribution**

$$p(\mathbf{x} \mid \boldsymbol{\theta}, \boldsymbol{\varphi}) = h(x, \boldsymbol{\varphi}) \exp\left\{ \frac{\boldsymbol{\theta}^T \mathbf{x} - A(\boldsymbol{\theta})}{a(\boldsymbol{\varphi})} \right\}$$

- **Leads to a simple gradient form**
- **Example: Bernoulli distribution**

$$p(x \mid \mu) = \mu^x (1-\mu)^{1-x} = \exp\left\{ \log\left(\frac{\mu}{1-\mu} \right) x + \log(1-\mu) \right\}$$

$$\theta = \log\left(\frac{\mu}{1-\mu} \right) \qquad \mu = \frac{1}{1+e^{-\theta}}$$

  - **Logistic function matches the Bernoulli**

Non-linear extension of logistic regression

- use **feature (basis) functions** to model **nonlinearities**
  - the same trick as used for the linear regression

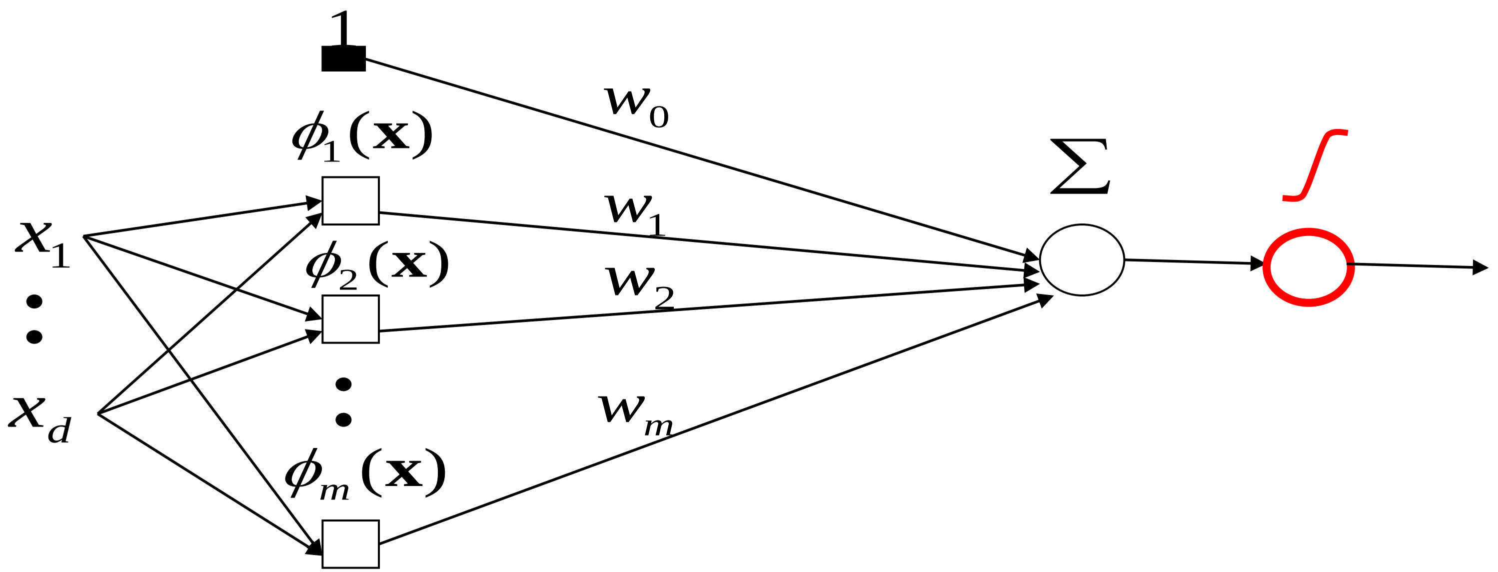
Linear regression

$$f(\mathbf{x}) = w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x})$$

Logistic regression

$$f(\mathbf{x}) = g(w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x}))$$

$\phi_j(\mathbf{x})$ - an arbitrary function of $\mathbf{x}$



Regularization

Similarly to the linear regression we can penalize the logistic regression or other GLM models for their complexity

- **L1 (lasso) regularization penalty**
- **L2 (ridge) regularization penalty**

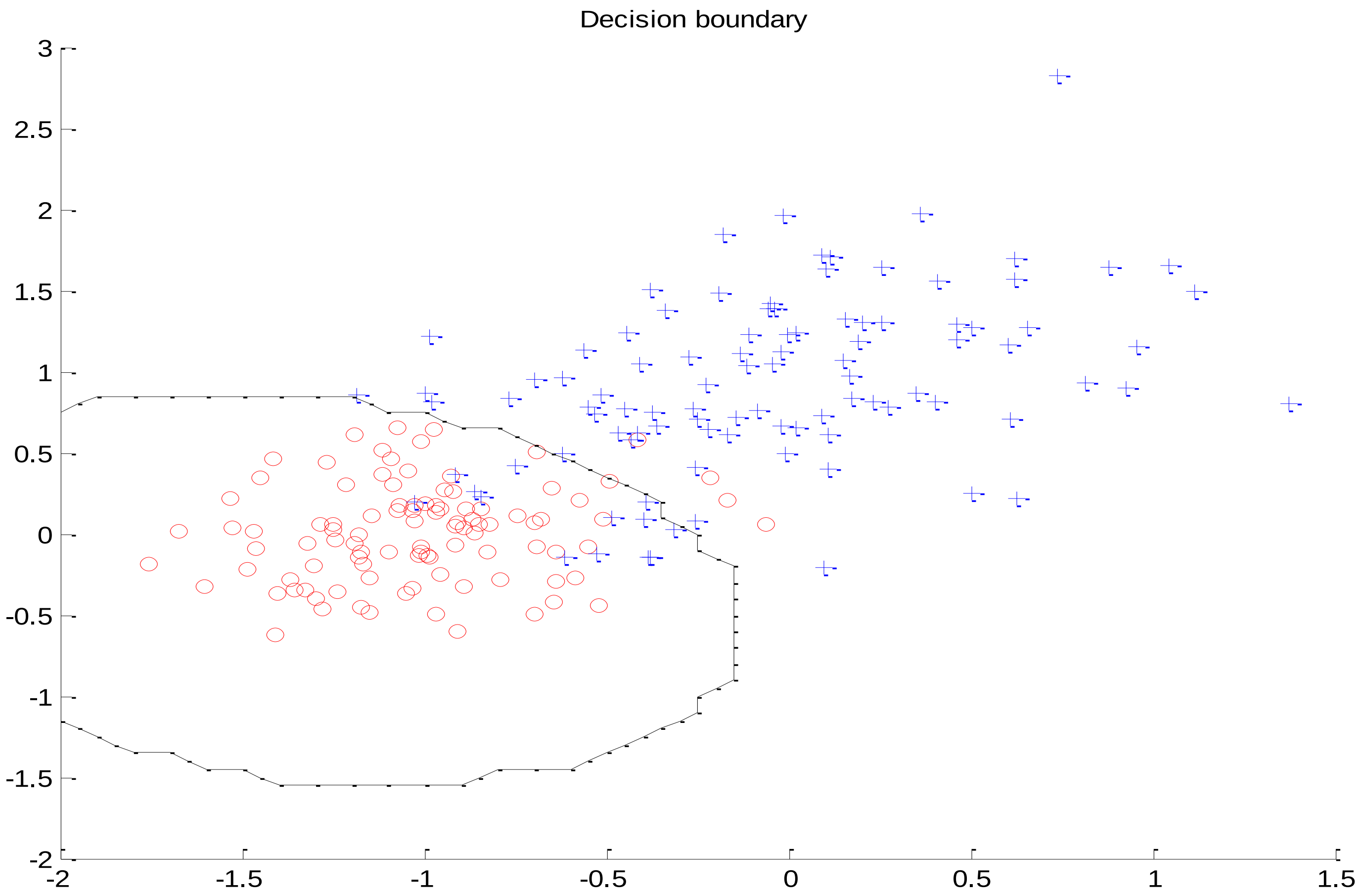
- Typically: the optimization of weights **w** looks as follows

$$\min_{\mathbf{w}} \quad \underbrace{Loss(D,\mathbf{w})}_{\text{fit}} + \underbrace{Q(\mathbf{w})}_{\text{Complexity penalty}}$$

- $Loss(D,\mathbf{w})$ functions:
  - Mean squared error
  - Negative log-likelihood
- Regularization penalty $Q(\mathbf{w})$: L1, L2 or a combination

When does the logistic regression fail?

- Quadratic decision boundary is needed



When does the logistic regression fail?

- Another example of a non-linear decision boundary