First-order logic

Limitations of propositional logic

World we want to represent and reason about consists of a number of objects with variety of properties and relations among them

Propositional logic:
• Represents statements about the world without reflecting this structure and without modeling these entities explicitly

Consequence:
• some knowledge is hard or impossible to encode in the propositional logic.
• Two cases that are hard to represent:
  – Statements about similar objects, relations
  – Statements referring to groups of objects.
Limitations of propositional logic

- Statements about similar objects and relations needs to be enumerated
- Example: Seniority of people domain

Assume we have:  
John is older than Mary  
Mary is older than Paul

To derive John is older than Paul we need:  
John is older than Mary \land Mary is older than Paul  
⇒ John is older than Paul

Assume we add another fact: Jane is older than Mary

To derive Jane is older than Paul we need:  
Jane is older than Mary \land Mary is older than Paul  
⇒ Jane is older than Paul

What is the problem?

Problem: KB grows large
Limitations of propositional logic

- **Statements about similar objects and relations needs to be enumerated**
- **Example:** Seniority of people domain
  For inferences we need:
  
  \[
  \begin{align*}
  \text{John is older than Mary} & \land \text{Mary is older than Paul} \\
  \Rightarrow & \quad \text{John is older than Paul} \\
  \text{Jane is older than Mary} & \land \text{Mary is older than Paul} \\
  \Rightarrow & \quad \text{Jane is older than Paul}
  \end{align*}
  \]

- **Problem:** if we have many people and facts about their seniority we need represent many rules like this to allow inferences
- **Possible solution:** introduce variables

\[
\begin{align*}
\text{Pers}_A \text{ is older than Pers}_B & \land \text{Pers}_B \text{ is older than Pers}_C \\
\Rightarrow & \quad \text{Pers}_A \text{ is older than Pers}_C
\end{align*}
\]
Limitations of propositional logic

• **Statements referring to groups of objects require exhaustive enumeration of objects**

• **Example:**

  Assume we want to express *Every student likes vacation*

  Doing this in propositional logic would require to include statements about every student

  \[\text{John likes vacation} \land \text{Mary likes vacation} \land \text{Ann likes vacation} \land \cdots\]

• **Solution:** Allow quantification in statements
First-order logic (FOL)

- More expressive than propositional logic

- Eliminates deficiencies of PL by:
  - Representing objects, their properties, relations and statements about them;
  - Introducing variables that refer to an arbitrary objects and can be substituted by a specific object
  - Introducing quantifiers allowing us to make statements over groups objects without the need to represent each of them separately

Logic

Logic is defined by:

- A set of sentences
  - A sentence is constructed from a set of primitives according to syntax rules.

- A set of interpretations
  - An interpretation gives a semantic to primitives. It associates primitives with objects, values in the real world.

- The valuation (meaning) function $V$
  - Assigns a truth value to a given sentence under some interpretation

\[ V : \text{sentence} \times \text{interpretation} \rightarrow \{ \text{True}, \text{False} \} \]
First-order logic. Syntax.

**Term** - syntactic entity for representing objects

**Terms in FOL:**
- **Constant symbols:** represent specific objects
  - E.g. John, France, car89
- **Variables:** represent objects of a certain type (type = domain of discourse)
  - E.g. x,y,z
- **Functions** applied to one or more terms
  - E.g. father-of(John)
    
    \[
    \text{father-of(father-of(John))}
    \]

First order logic. Syntax.

**Sentences in FOL:**
- **Atomic sentences:**
  - A **predicate symbol** applied to 0 or more terms

  **Examples:**
  
  Red(car12),
  
  Sister(Amy, Jane);
  
  Manager(father-of(John));

  - \( t_1 = t_2 \) **equivalence** of terms

  **Example:**
  
  \( John = \text{father-of}(Peter) \)
First order logic. Syntax.

Sentences in FOL:
• **Complex sentences:**
  - Assume $\phi$, $\psi$ are sentences in FOL. Then:
    - $(\phi \land \psi)$ (\phi \lor \psi) (\phi \Rightarrow \psi) (\phi \Leftrightarrow \psi) \neg \psi$
    and
  - $\forall x \phi$ $\exists y \phi$
    are sentences

Symbols $\exists$, $\forall$
- stand for the **existential** and the **universal** quantifier

Semantics. Interpretation.

An interpretation $I$ is defined by a **mapping** to the domain of discourse $D$ or relations on $D$
• **domain of discourse:** a set of objects in the world we represent and refer to;

An interpretation $I$ maps:
• Constant symbols to objects in $D$
  
  $I(John) = \text{John}$
  
  Predicate symbols to relations, properties on $D$
  
  $I(\text{brother}) = \{ \langle \text{John}, \text{John} \rangle; \langle \text{John}, \text{Jane} \rangle; \ldots \}$
  
  Function symbols to functional relations on $D$
  
  $I(\text{father-of}) = \{ \langle \text{John}, \text{Jane} \rangle \rightarrow \text{Jane}; \langle \text{John}, \text{Jane} \rangle \rightarrow \text{Jane}; \ldots \}$
Semantics of sentences

Meaning (evaluation) function:

\[ V : \text{sentence} \times \text{interpretation} \rightarrow \{ \text{True} , \text{False} \} \]

A predicate \[ \text{predicate}(\text{term-1, term-2, term-3, term-n}) \] is true for the interpretation \( I \), iff the objects referred to by \( \text{term-1, term-2, term-3, term-n} \) are in the relation referred to by \( \text{predicate} \)

\[ I(\text{John}) = \begin{array}{c}
\text{\( I(Paul) = \)} \\
\end{array} \]

\[ I(\text{brother}) = \begin{Bmatrix}
\langle \text{\( 1 \)} \text{\( 2 \)} \text{\( 3 \)} \rangle ; \langle \text{\( 4 \)} \text{\( 5 \)} \text{\( 6 \)} \rangle ; \ldots \\
\end{Bmatrix} \]

\[ I(\text{\( \text{brother} \) (John, Paul)} = \langle \text{\( 1 \)} \text{\( 2 \)} \text{\( 3 \)} \rangle \quad \text{in} \quad I(\text{brother}) \]

\[ V(\text{\( \text{brother} \) (John, Paul), I} = \text{True} \]

Semantics of sentences.

- **Equality**
  \[ V(\text{term-1} = \text{term-2}, I) = \text{True} \]
  Iff \( I(\text{term-1}) = I(\text{term-2}) \)

- **Boolean expressions:** standard
  E.g. \[ V(\text{sentence-1} \lor \text{sentence-2}, I) = \text{True} \]
  Iff \( V(\text{sentence-1}, I) = \text{True} \) or \( V(\text{sentence-2}, I) = \text{True} \)

- **Quantifications**
  \[ V(\forall x \phi, I) = \text{True} \]
  Iff for all \( d \in D \) \( V(\phi, I[x/d]) = \text{True} \)

  \[ V(\exists x \phi, I) = \text{True} \]
  Iff there is a \( d \in D \), s.t. \( V(\phi, I[x/d]) = \text{True} \)
Sentences with quantifiers

- **Universal quantification**

  \[ \forall x \text{ smart}(x) \]

- Assume the universe of discourse of \( x \) are Upitt students

- Assume the universe of discourse of \( x \) are students

  \[ \forall x \text{ at}(x, \text{Upitt}) \Rightarrow \text{smart}(x) \]

- Assume the universe of discourse of \( x \) are people

  \[ \forall x \text{ student}(x) \land \text{at}(x, \text{Upitt}) \Rightarrow \text{smart}(x) \]

**Typically the universal quantifier connects with implication**
Sentences with quantifiers

• **Existential quantification**

  *Someone at CMU is smart*

  • Assume the universe of discourse of x are CMU affiliates
    \[ \exists x \text{ smart}(x) \]

  • Assume the universe of discourse of x are people
    \[ \exists x \text{ at}(x, \text{CMU}) \land \text{smart}(x) \]

  *Typically the existential quantifier connects with a conjunction*
Translation with quantifiers

• Assume two predicates S(x) and P(x)

Universal statements typically tie with implications
• All S(x) is P(x)
  – $\forall x ( S(x) \rightarrow P(x) )$
• No S(x) is P(x)
  – $\forall x ( S(x) \rightarrow \neg P(x) )$

Existential statements typically tie with conjunctions
• Some S(x) is P(x)
  – $\exists x ( S(x) \land P(x) )$
• Some S(x) is not P(x)
  – $\exists x ( S(x) \land \neg P(x) )$

Nested quantifiers

• More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:
• There is a person who loves everybody.
• Translation:
  – Assume:
    • Variables x and y denote people
    • A predicate L(x,y) denotes: “x loves y”
  • Then we can write in the predicate logic:
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- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:
- There is a person who loves everybody.

Translation:
- Assume:
  - Variables x and y denote people
  - A predicate L(x,y) denotes “x loves y”
- Then we can write in the predicate logic:
  \[ \exists x \forall y L(x,y) \]

Translation exercise

Suppose:
- Variables x,y denote people
- L(x,y) denotes “x loves y”.

Translate:
- Everybody loves Raymond. \[ \forall x L(x,\text{Raymond}) \]
- Everybody loves somebody. \[ \forall x \exists y L(x,y) \]
- There is somebody whom everybody loves. \[ \exists y \forall x L(x,y) \]
- There is somebody who Raymond doesn't love. \[ \exists y \neg L(\text{Raymond},y) \]
- There is somebody whom no one loves. \[ \exists y \forall x \neg L(x,y) \]