Decision making in the presence of uncertainty II

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Information-gathering actions

- Many actions and their outcomes irreversibly change the world
- **Information-gathering (exploratory) actions:**
  - make an inquiry about the world
  - **Key benefit:** reduction in the uncertainty
- **Example:** medicine
  - Assume a patient is admitted to the hospital with some set of initial complaints
  - We are uncertain about the underlying problem and consider a surgery, or a medication to treat them
  - But there are often lab tests or observations that can help us to determine more closely the disease the patient suffers from
  - **Goal of lab tests:** Reduce the uncertainty of outcomes of treatments so that better treatment option can be chosen
**Decision-making with exploratory actions**

In decision trees:

- **Exploratory actions** can be represented and reasoned about the same way as other actions.

How do we capture the effect of exploratory actions in the decision tree model?

- Information obtained through exploratory actions may affect the probabilities of later outcomes
  - Recall that the probabilities on later outcomes can be conditioned on past observed outcomes and past actions
  - Sequence of past actions and outcomes is “remembered” within the decision tree branch

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**Oil wildcatter problem.**

An oil wildcatter has to make a decision of whether to drill or not to drill on a specific site

- **Chance of hitting an oil deposit:**
  - Oil: 40% \( P(Oil = T) = 0.4 \)
  - No-oil: 60% \( P(Oil = F) = 0.6 \)
- **Cost of drilling:** 70K
- **Payoffs:**
  - Oil: 220K
  - No-oil: 0 K

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[

Drill

0.4

0.6

220-70=150

-70

No-drill

1.0

0

]
Oil wildcatter problem.

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- **Payoffs:**
  - Oil: 220K
  - No-oil: 0 K

![Decision Tree Diagram]

Oil wildcatter problem

- Assume that in addition to the drill/no-drill choices we have an option to run the **seismic resonance test**
- **Seismic resonance test results:**
  - **Closed pattern** (more likely when the hole holds the oil)
  - **Diffuse pattern** (more likely when empty)

\[
P(Oil \mid \text{Seismic resonance test})
\]

<table>
<thead>
<tr>
<th>Seismic resonance test pattern</th>
<th>closed</th>
<th>diffuse</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Oil</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>True</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>False</td>
<td>0.3</td>
<td>0.7</td>
</tr>
</tbody>
</table>

- **Test cost:** 10K
Oil wildcatter problem.

- **Decision tree**

```
  Test 0.5
    | 0.5
    | Drill 18 0.4 (closed) Drilling 220-70=150
    |   0.6 (diffuse) No-drill
    |   0
    |   1.0
    |   0
No-drill 0.16 Drill 18 0.6 (no-oil) Drilling 220-70=140
    0.36 (no-oil) No-drill
    1.0
    0
```

- **Alternative model**

```
  NoTest 0.5
    | 0.5
    | Drill 18 0.4 (closed) Drilling 220-70=150
    |   0.6 (diffuse) No-drill
    |   0
    |   1.0
    |   0
```

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Oil wildcatter problem.

- Decision tree probabilities

\[
P(Oil \mid Test = closed) = \frac{P(Test = closed \mid Oil = T)P(Oil = T)}{P(Test = closed)}
\]

\[
P(Oil = F \mid Test = closed) = \frac{P(Test = closed \mid Oil = F)P(Oil = F)}{P(T = closed)}
\]

\[
P(Test = closed) = P(Test = closed \mid Oil = F)P(Oil = F) + P(Test = closed \mid Oil = T)P(Oil = T)
\]

Oil wildcatter problem.

- Decision tree probabilities

\[
P(\text{Test}) = P(\text{Test} = \text{closed} \mid Oil = F)P(Oil = F) + P(\text{Test} = \text{diff} \mid Oil = T)P(Oil = T)
\]

\[
P(\text{Test} = \text{diff}) = P(\text{Test} = \text{diff} \mid Oil = F)P(Oil = F) + P(\text{Test} = \text{diff} \mid Oil = T)P(Oil = T)
\]
Oil wildcatter problem.

• Decision tree

The presence of the test and its result affected our decision:

if test = closed then drill
if test = diffuse then do not drill
Value of information

- When the test makes sense?
- Only when its result makes the decision maker to change his mind, that is he decides not to drill.

- Value of information:
  - Measure of the goodness of the information from the test
  - Difference between the expected value with and without the test information

- Oil wildcatter example:
  - Expected value without the test = 18
  - Expected value with the test = 25.4
  - Value of information for the seismic test = 7.4

Utilities

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Selection based on expected values

- **Until now:** The optimal action choice was the option that maximized the expected monetary value.
- **But is the expected monetary value always the quantity we want to optimize?**

![Diagram of investment options]

- **Stock 1**
  - (up) 110
  - (down) 90

- **Stock 2**
  - (up) 140
  - (down) 80

- **Bank**
  - 101

- **Home**
  - 100

**Is the expected monetary value always the quantity we want to optimize?**

- **Answer:** Yes, but only if we are risk-neutral.
- But what if we do not like the risk (we are risk-averse)?
- In that case we may want to get the premium for undertaking the risk (of loosing the money)
- **Example:**
  - we may prefer to get $101 for sure against $102 in expectation but with the risk of loosing the money
- **Problem:** How to model decisions and account for the risk?
- **Solution:** use utility function, and utility theory
Utility function

• **Utility function (denoted U)**
  – Quantifies how we “value” outcomes, i.e., it reflects our preferences
  – Can be also applied to “value” outcomes other than money and gains (e.g. utility of a patient being healthy, or ill)

• **Decision making:**
  – uses expected utilities (denoted EU)

\[
EU(X) = \sum_{x \in \Omega_X} P(X = x) U(X = x)
\]

\[U(X = x)\] the utility of outcome x

**Important !!!**

• Under some conditions on preferences we can always design the utility function that fits our preferences

Utility theory

• Defines axioms on preferences that involve uncertainty and ways to manipulate them.
• Uncertainty is modeled through **lotteries**
  – **Lottery:**
    \[
    [ p : A ; (1 - p) : C ]
    \]
  • Outcome A with probability p
  • Outcome C with probability (1-p)

• The following six constraints are known as the axioms of utility theory. The axioms are the most obvious semantic constraints on preferences with lotteries.

• **Notation:**
  \[\succ\] - preferable
  \[\sim\] - indifferent (equally preferable)
Axioms of the utility theory

- **Orderability**: Given any two states, the a rational agent prefers one of them, else the two as equally preferable.
  \[(A \succ B) \lor (B \succ A) \lor (A \sim B)\]

- **Transitivity**: Given any three states, if an agent prefers \(A\) to \(B\) and prefers \(B\) to \(C\), agent must prefer \(A\) to \(C\).
  \[(A \succ B) \land (B \succ C) \Rightarrow (A \succ C)\]

- **Continuity**: If some state \(B\) is between \(A\) and \(C\) in preference, then there is a \(p\) for which the rational agent will be indifferent between state \(B\) and the lottery in which \(A\) comes with probability \(p\), \(C\) with probability \((1-p)\).
  \[(A \succ B \succ C) \Rightarrow \exists p \, [p : A; (1-p) : C] \sim B\]

Axioms of the utility theory

- **Substitutability**: If an agent is indifferent between two lotteries, \(A\) and \(B\), then there is a more complex lottery in which \(A\) can be substituted with \(B\).
  \[(A \sim B) \Rightarrow [p : A;(1-p) : C] \sim [p : B;(1-p) : C]\]

- **Monotonicity**: If an agent prefers \(A\) to \(B\), then the agent must prefer the lottery in which \(A\) occurs with a higher probability
  \[(A \succ B) \Rightarrow (p > q \iff [p : A;(1-p) : B] \succ [q : A;(1-q) : B])\]

- **Decomposability**: Compound lotteries can be reduced to simpler lotteries using the laws of probability.
  \[[p : A;(1-p) : [q : B;(1-q) : C]] \Rightarrow [p : A;(1-p)q : B;(1-p)(1-q) : C]\]
Utility theory

If the agent obeys the axioms of the utility theory, then

1. there exists a real valued function $U$ such that:
   $$ U(A) > U(B) \iff A > B $$
   $$ U(A) = U(B) \iff A \sim B $$

2. The utility of the lottery is the expected utility, that is the sum of utilities of outcomes weighted by their probability
   $$ U[p : A; (1 - p) : B] = pU(A) + (1 - p)U(B) $$

3. Rational agent makes the decisions in the presence of uncertainty by maximizing its expected utility

Utility functions

We can design a utility function that fits our preferences if they satisfy the axioms of utility theory.

- But how to design the utility function for monetary values so that they incorporate the risk?
- What is the relation between utility function and monetary values?

- Assume we lose or gain $1000.
  - Typically this difference is more significant for lower values (around $100 - 1000) than for higher values (~$1,000,000)
- What is the relation between utilities and monetary value for a typical person?
Utility functions

• What is the relation between utilities and monetary value for a typical person?
• Concave function that flattens at higher monetary values

Utility functions

• Expected utility of a sure outcome of 750 is 750
Utility functions

Assume a lottery $L$ \([0.5: 500, 0.5:1000]\)
- Expected value of the lottery = 750
- Expected utility of the lottery $EU(L)$ is different:
  - $EU(L) = 0.5U(500) + 0.5*U(1000)$

Risk aversion – a bonus is required for undertaking the risk

Utility functions

- Expected utility of the lottery $EU(lottery\ L) < EU(sure\ 750)$

Risk aversion – a bonus is required for undertaking the risk