CS 2740 Knowledge Representation
Lecture 6

# Propositional logic. Horn clauses

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## Logical inference problem

**Logical inference problem**:
- **Given:**
  - a knowledge base KB (a set of sentences) and
  - a sentence $\alpha$ (called **a theorem)**,
- **Does a KB semantically entail** $\alpha$ ? $KB \models \alpha$ ?

In other words: In all interpretations in which sentences in the KB are true, is also $\alpha$ true?

## Solving logical inference problem

In the following:

**How to design the procedure that answers:**

$$KB \models \alpha \ ?$$

**Three approaches:**

- **Truth-table approach**
- **Inference rules**
- **Conversion to the inverse SAT problem**
  - **Resolution-refutation**



## KB in restricted forms

If the sentences in the KB are restricted to some special forms some of the sound inference rules may become complete

**Example:**

- **Horn form (Horn normal form)**

$$(A \vee \neg B) \wedge (\neg A \vee \neg C \vee D)$$

Can be written also as: $(B \Rightarrow A) \wedge ((A \wedge C) \Rightarrow D)$

- **Two inference rules that are sound and complete with respect to propositional symbols for KBs in the Horn normal form:**
  - **Resolution (positive unit resolution)**
  - **Modus ponens**

# KB in Horn form

- **Horn form:** a clause with **at most one positive literal**

$$(A \vee \neg B) \wedge (\neg A \vee \neg C \vee D)$$

- **Not all sentences in propositional logic can be converted into the Horn form**
- **KB in Horn normal form**:
  - Two types of propositional statements:
    - **Rules** $(\neg B_1 \vee \neg B_2 \vee \dots \neg B_k \vee A)$

$$(\neg(B_1 \wedge B_2 \wedge \dots B_k) \vee A)$$

$$(B_1 \wedge B_2 \wedge \dots B_k \Rightarrow A)$$

    - Propositional symbols: **facts** $B$



# KB in Horn form

- **Application of the modus ponens:**
  - Infers new facts from previous facts

$$\frac{B \Rightarrow A, \quad B}{A}$$

$$\frac{(B_1 \wedge B_2 \wedge \dots B_k \Rightarrow A), B_1, B_2, \dots B_k}{A}$$

  - Modus ponens is **sound and complete with respect to propositional symbols** for the KBs in the Horn normal form

# Complexity of inferences for KBs in HNF

**Question:**

**How efficient the inferences in HNF can be?**

**Answer:**

**Procedures linear in the size of the KB in the Horn form exist.**

- Size of a clause: the number of literals it contains.
- Size of the KB in the HNF: the sum of the sizes of its elements.

**Example:**

$$A, B, (A \wedge B \Rightarrow C), (C \Rightarrow D), (C \Rightarrow E), (E \wedge F \Rightarrow G)$$

or

$$A, B, (\neg A \vee \neg B \vee C), (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$$

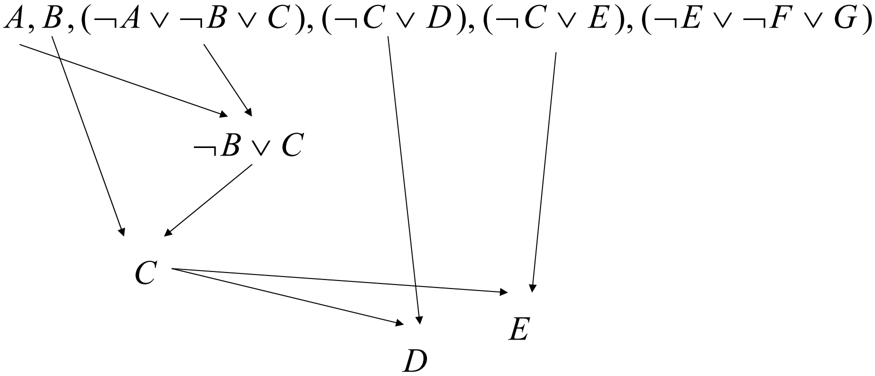
The size is: 12



# Complexity of inferences for KBs in HNF

How to do the inference? If the HNF (is in clausal form) we can apply resolution.

$$A, B, (\neg A \vee \neg B \vee C), (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$$

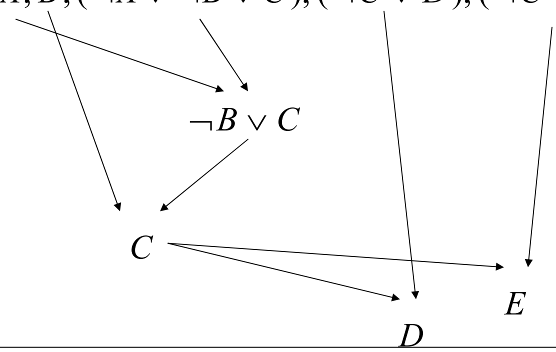


# Complexity of inferences for KBs in HNF

**Features:**

- Every resolution is a **positive unit resolution**; that is, a resolution in which **one clause is a positive unit clause** (i.e., a proposition letter).

$$A, B, (\neg A \vee \neg B \vee C), (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$$



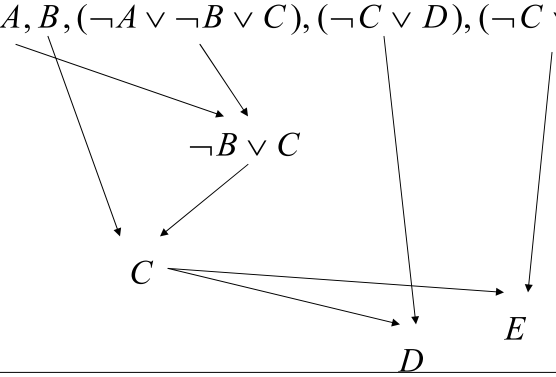



# Complexity of inferences for KBs in HNF

**Features:**

- At each resolution, the input clause which is not a unit clause is a logical consequence of the result of the resolution. (Thus, the input clause may be deleted upon completion of the resolution operation.)

$$A, B, (\neg A \vee \neg B \vee C), (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$$



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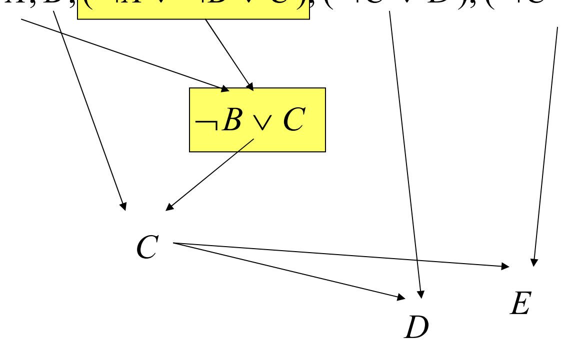


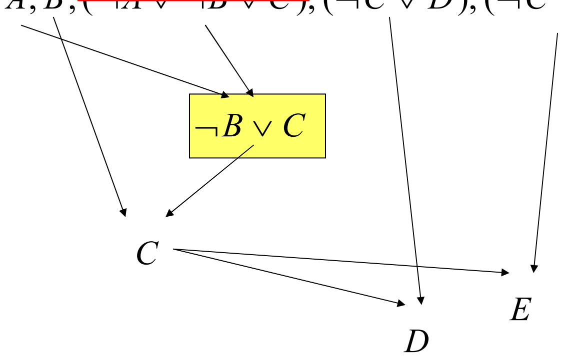



# Complexity of inferences for KBs in HNF

**Features:**

- Following this deletion, the size of the KB (the sum of the lengths of the remaining clauses) is one less than it was before the operation.)

$A, B,$ ~~$(\neg A \vee \neg B \vee C)$~~$, (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$

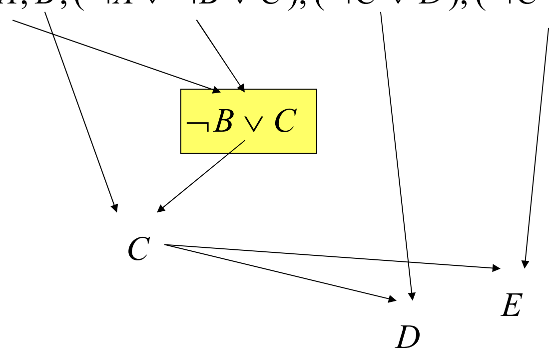


# Complexity of inferences for KBs in HNF

**Features:**

- If n is the size of the KB, then at most n positive unit resolutions may be performed on it.

$A, B,$ ~~$(\neg A \vee \neg B \vee C)$~~$, (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$






# Complexity of inferences for KBs in HNF

**Linear time algorithm:**

- **The number of positive unit resolutions is limited to the size of the formula (n)**
- **But to assure overall linear time we need to access each proposition in a constant time:**
- Data structures indexed by proposition names may be accessed in constant time. (This is possible if the proposition names are number in a range (e.g., 1..n), so that array lookup is the access operation.
- If propositions are accessed by name, then a symbol table is necessary, and the algorithm will run in time $O(n \cdot \log(n))$.

## Forward and backward chaining

Two inference procedures based on **modus ponens** for **Horn KBs**:

- **Forward chaining**

  **Idea:** Whenever the premises of a rule are satisfied, infer the conclusion. Continue with rules that became satisfied.

- **Backward chaining (goal reduction)**

  **Idea:** To prove the fact that appears in the conclusion of a rule prove the premises of the rule. Continue recursively.

Both procedures are **complete for KBs in the Horn form** !!!



## Forward chaining example

- **Forward chaining**

  **Idea:** Whenever the premises of a rule are satisfied, infer the conclusion. Continue with rules that became satisfied.

Assume the KB with the following rules and facts:

KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

R3: $C \wedge F \Rightarrow G$

---

F1: $A$

F2: $B$

F3: $D$

Theorem: $E$ ?

## Forward chaining example

**Theorem:** $E$

KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

R3: $C \wedge F \Rightarrow G$

---

F1: $A$

F2: $B$

F3: $D$



## Forward chaining example

**Theorem:** $E$

KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

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F1: $A$

F2: $B$

F3: $D$

**Rule R1 is satisfied.**

F4: $C$

# Forward chaining example

**Theorem:** $E$

KB: R1: $A \wedge B \Rightarrow C$

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---

F1: $A$
F2: $B$
F3: $D$

**Rule R1 is satisfied.**

F4: $C$

**Rule R2 is satisfied.**

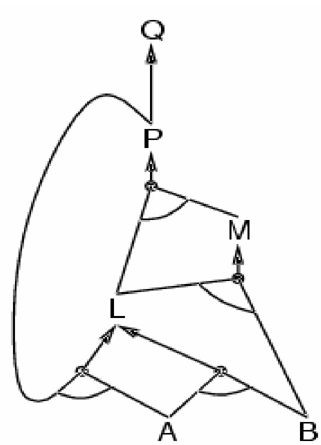
F5: $E$ ✓



# Forward chaining

- Efficient implementation: linear in the size of the KB
- **Example:**

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$



# Forward chaining

- Runs in time linear in the number of literals in the Horn formulae

```
function PL-FC-ENTAILS?(KB, q) returns true or false
  local variables: count, a table, indexed by clause, initially the number of premises
                   inferred, a table, indexed by symbol, each entry initially false
                   agenda, a list of symbols, initially the symbols known to be true

  while agenda is not empty do
      p ← POP(agenda)
      unless inferred[p] do
          inferred[p] ← true
          for each Horn clause c in whose premise p appears do
              decrement count[c]
              if count[c] = 0 then do
                  if HEAD[c] = q then return true
                  PUSH(HEAD[c], agenda)
  return false
```



# Forward chaining

- 

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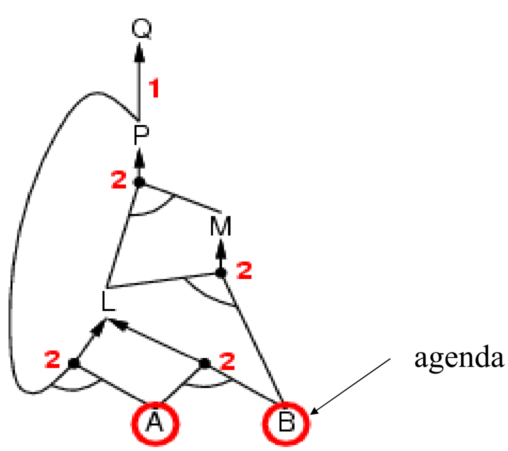

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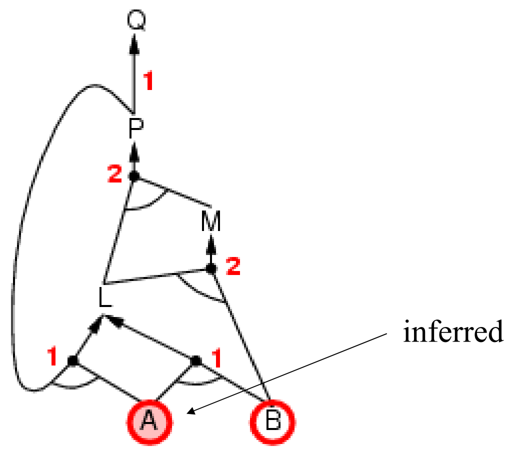




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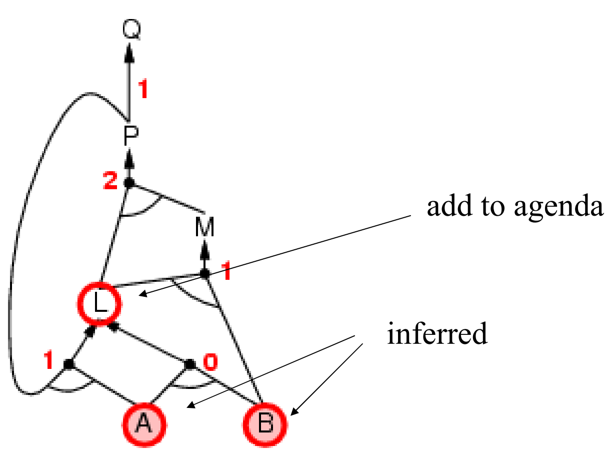

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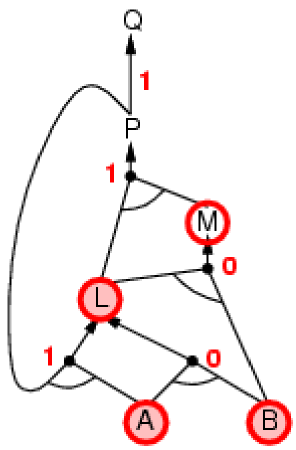




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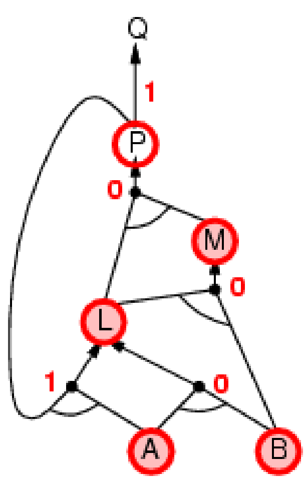

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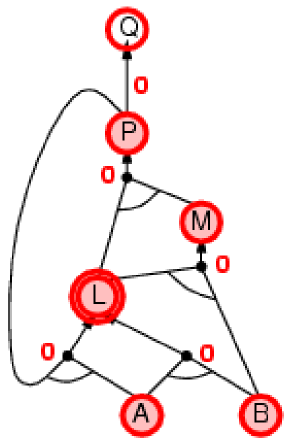
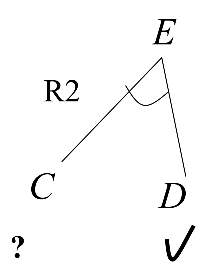





# Backward chaining example

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F1: $A$

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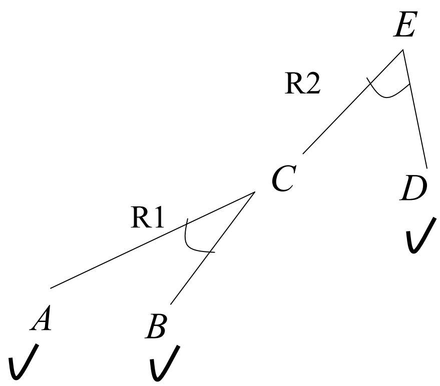
- Backward chaining is more focused:
  - tries to prove the theorem only

# Backward chaining example

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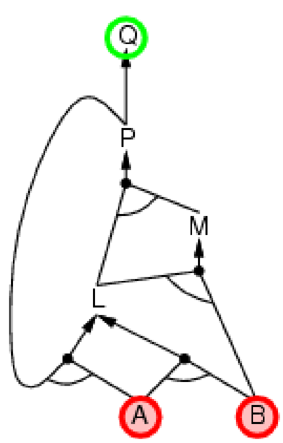

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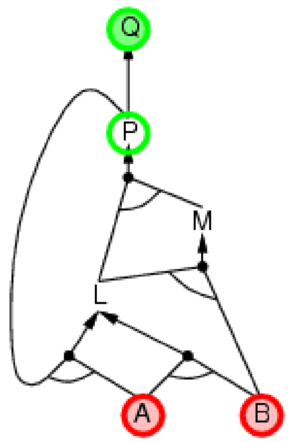




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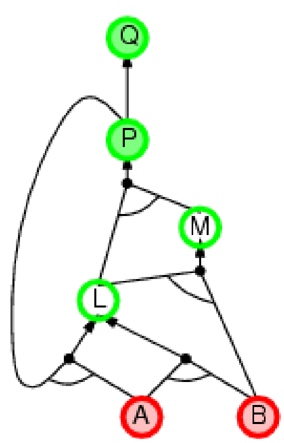

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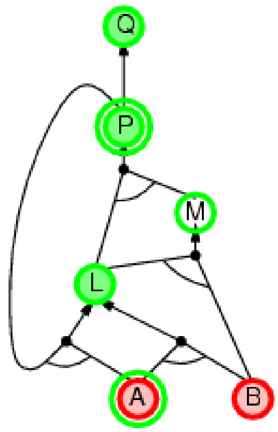




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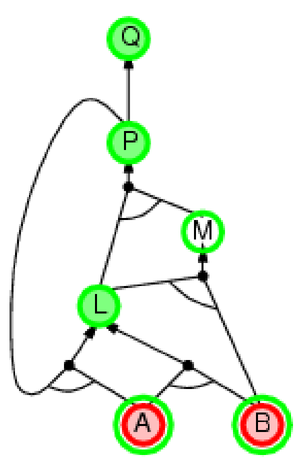

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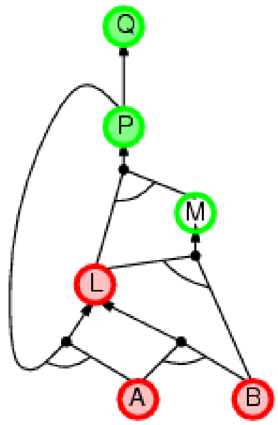




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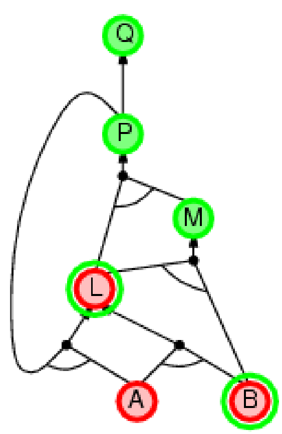

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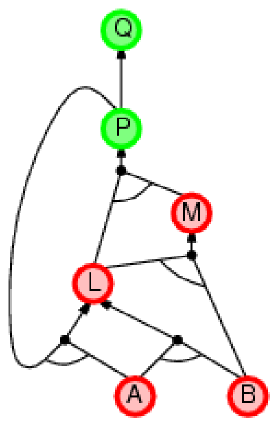




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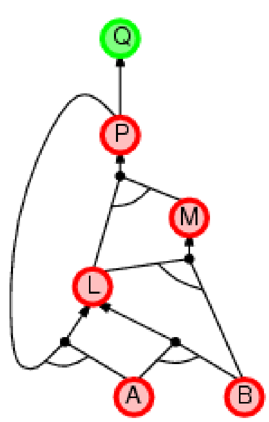

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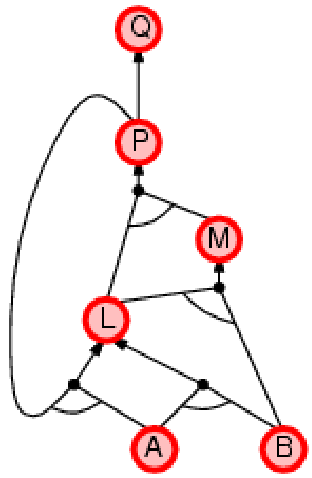




# Forward vs Backward chaining

- **FC is data-driven**, automatic, unconscious processing,
  – e.g., object recognition, routine decisions

- May do lots of work that is irrelevant to the goal

- **BC is goal-driven**, appropriate for problem-solving,
  – e.g., Where are my keys? How do I get into a PhD program?
- Complexity of BC can be **much less** than **linear in size of KB**

# KB agents based on propositional logic

- Propositional logic allows us to build **knowledge-based agents** capable of answering queries about the world by infering new facts from the known ones
- **Example:** an agent for diagnosis of a bacterial disease

**Facts:** The stain of the organism is gram-positive
The growth conformation of the organism is chains

**Rules:** **(If)** The stain of the organism is gram-positive $\wedge$
The morphology of the organism is coccus $\wedge$
The growth conformation of the organism is chains
**(Then)** $\Rightarrow$ The identity of the organism is streptococcus



# Example. Animal identification system.

I1. If the animal has hair then it is a mammal
I2. If the animal gives milk then it is a mammal
I3. If the animal has feathers then it is a bird
I4. If the animal flies and it lays eggs then it is a bird
I5. If the animal is a mammal and it eats meat then it is a carnivore
I6. If the animal is a mammal and it has pointed teeth and it has claws and its eyes point forward then it is a carnivore
I7. If the animal is a mammal and it has hoofs then it is an ungulate
18. If the animal is a mammal and it chews cud then it is an ungulate and it is even-toed
I9. If the animal is a carnivore and it has a tawny color and it has dark spots then it is a cheetah
I10. If the animal is a carnivore and it has a tawny color and it has black strips then it is a tiger
I11. If the animal is an ungulate and it has long legs and it has a long neck and it has a tawny color and it has dark spots then it is a giraffe
I12. If the animal is an ungulate and it has a white color and it has black stripes then it is a zebra
Il3. If the animal is a bird and it does not fly and it has long legs and it has a long neck and it is black and white then it is an ostrich,
Il4. If the animal is a bird and it does not fly and it swims and it is black and white then it is a penguin
Il5. If the animal is a bird and it is a good flyer then it is an albatross.