CS 1675 Introduction to Machine Learning
Lecture 23

Reinforcement learning

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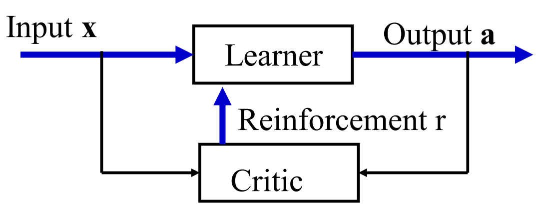
Reinforcement learning

- **We want to learn a control policy:** $\pi : X \rightarrow A$
- We see examples of **x** (but outputs *a* are not given)
- Instead of *a* we get a feedback *r* (reinforcement, reward) from a **critic** quantifying how good the selected output was



- The reinforcements may not be deterministic
- **Goal:** find $\pi : X \rightarrow A$ with the best expected reinforcements

Gambling example

- **Game:** 3 different biased coins are tossed
  - The coin to be tossed is selected randomly from the three options and I always see which coin I am going to play next
  - I make bets on head or tail and I always wage $1
  - If I win I get $1, otherwise I lose my bet
- **RL model:**
  - **Input:** X – a coin chosen for the next toss,
  - **Action:** A – choice of head or tail,
  - **Reinforcements:** {1, -1}
- **A policy** $\pi : X \to A$

Example: $\pi :$ | **Coin1 → *head*** | **Coin2 → *tail*** | **Coin3 → *head*** |

Gambling example

- **RL model:**
  - **Input:** X – a coin chosen for the next toss,
  - **Action:** A – choice of head or tail,
  - **Reinforcements:** {1, -1}
  - **A policy** $\pi :$ | **Coin1 → *head*** | **Coin2 → *tail*** | **Coin3 → *head*** |
- **Learning goal: find** $\pi : X \to A$ $\pi :$ | **Coin1 → ?** | **Coin2 → ?** | **Coin3 → ?** |

maximizing future expected profits

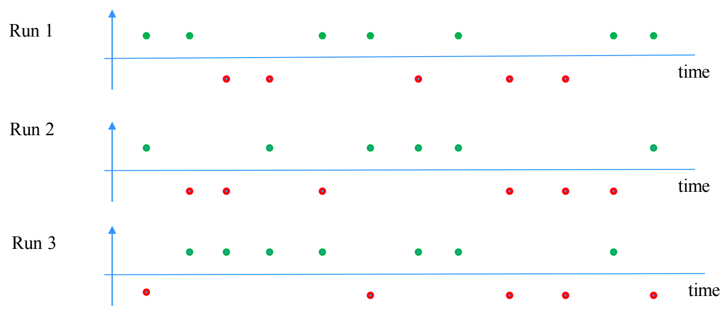
$$E(\sum_{t=0}^{\infty} \gamma^t r_t)$$

$0 \le \gamma < 1$

a discount factor = present value of money

Expected rewards

- **Expected rewards for** $\pi : X \rightarrow A$

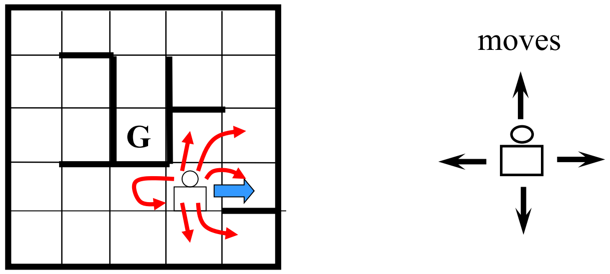


…

$E(\sum_{t=0}^{\infty} \gamma^t r_t)$ Expectation over many possible discounted reward trajectories for $\pi : X \rightarrow A$

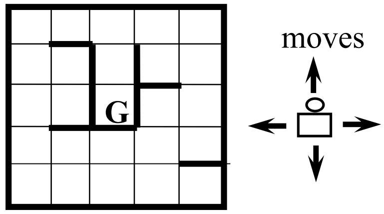
Agent navigation example

- **Agent navigation in the Maze**:
  - 4 moves in compass directions
  - Effects of moves are stochastic – we may wind up in other than intended location with a non-zero probability
  - **Objective:** learn how to reach the goal state in the shortest expected time



Agent navigation example

- **The RL model:**
  - **Input:** X – position of an agent
  - **Output:** A –a move
  - **Reinforcements:** R
    - -1 for each move
    - +100 for reaching the goal
  - **A policy:** $\pi : X \rightarrow A$



π :
Position 1 → ***right***
Position 2 → ***right***
…
Position 20 → ***left***

- **Goal: find the policy maximizing future expected rewards**

$$E\left(\sum_{t=0}^{\infty} \gamma^t r_t\right) \qquad 0 \le \gamma < 1$$

Objectives of RL learning

- **Objective:**
  Find a mapping $\pi^* : X \rightarrow A$
  That maximizes some combination of future reinforcements (rewards) received over time
- **Valuation models (quantify how good the mapping is):**
  - **Finite horizon model**

    $$E\left(\sum_{t=0}^{T} r_t\right) \qquad \text{Time horizon: } T > 0$$

  - **Infinite horizon discounted model**

    $$E\left(\sum_{t=0}^{\infty} \gamma^t r_t\right) \qquad \text{Discount factor: } 0 \le \gamma < 1$$

  - **Average reward**

    $$\lim_{T\to\infty} \frac{1}{T} E\left(\sum_{t=0}^{T} r_t\right)$$

Exploration vs. Exploitation

- The (learner) actively interacts with the environment:
  - At the beginning the learner does not know anything about the environment
  - It gradually gains the experience and learns how to react to the environment
- **Dilemma (exploration-exploitation):**
  - After some number of steps, should I select the best current choice (**exploitation**) or try to learn more about the environment (**exploration**)?
  - **Exploitation** may involve the selection of a sub-optimal action and prevent the learning of the optimal choice
  - **Exploration** may spend to much time on trying bad currently suboptimal actions

Effects of actions on the environment

Effect of actions on the environment (next input **x** to be seen)

- No effect, the distribution over possible **x** is fixed; action consequences (rewards) are seen immediately,
- Otherwise, distribution of **x** can change; the rewards related to the action can be seen with some delay.

Leads to two forms of **reinforcement learning:**

- **Learning with immediate rewards**
  - **Gambling example**
- **Learning with delayed rewards**
  - **Agent navigation example**; move choices affect the state of the environment (position changes), a big reward at the goal state is delayed

RL with immediate rewards

- **Game:** 3 different biased coins are tossed
  - The coin to be tossed is selected randomly from the three options and I always see which coin I am going to play next
  - I make bets on head or tail and I always wage $1
  - If I win I get $1, otherwise I lose my bet
- **RL model:**
  - **Input:** X – a coin chosen for the next toss
  - **Action:** A – head or tail bet
  - **Reinforcements:** {1, -1}
- **Learning goal: find** $\pi : X \rightarrow A$

  maximizing the future expected profits over time

  $E(\sum_{t=0}^{\infty} \gamma^t r_t)$ $\quad 0 \le \gamma < 1 \quad$ a discount factor

RL with immediate rewards

- **Expected reward**

  $E(\sum_{t=0}^{\infty} \gamma^t r_t)$ $\quad 0 \le \gamma < 1$
- **Immediate reward case:**
  - Reward for the choice becomes available immediately
  - Our action does not affect the environment and thus future rewards

    $$E(\sum_{t=0}^{\infty} \gamma^t r_t) = E(r_0) + E(\gamma r_1) + E(\gamma^2 r_2) + \ldots$$

    $r_0, r_1, r_2 \cdots$ Rewards for every step
  - Expected one step reward for input **x** and the choice *a* : $R(\mathbf{x}, a)$

RL with immediate rewards

Immediate reward case:

- Reward for the choice a becomes available immediately
- **Expected reward for the input x and choice a:** $R(\mathbf{x}, a)$
  - For the gambling problem it is:

$$R(\mathbf{x}, a_i) = \sum_j r(\omega_j \mid a_i, \mathbf{x}) P(\omega_j \mid \mathbf{x}, a_i)$$

  - ω_j- a future outcome of the coin toss
  - Recall the definition of the expected loss
- **Expected one step reward for a strategy** $\pi : X \rightarrow A$

$$R(\pi) = \sum_x R(\mathbf{x}, \pi(\mathbf{x})) P(\mathbf{x})$$

$R(\pi)$ is the expected reward for $r_0, r_1, r_2 \ldots$

RL with immediate rewards

- **Expected reward**

$$E(\sum_{t=0}^{\infty} \gamma^t r_t) = E(r_0) + E(\gamma r_1) + E(\gamma^2 r_2) + \ldots$$

- **Optimizing the expected reward :**

$$\max_\pi E(\sum_{t=0}^{\infty} \gamma^t r_t) = \max_\pi \sum_{t=0}^{\infty} \gamma^t E(r_t) = \max_\pi \sum_{t=0}^{\infty} \gamma^t R(\pi) = \max_\pi R(\pi)(\sum_{t=0}^{\infty} \gamma^t)$$

$$= (\sum_{t=0}^{\infty} \gamma^t) \max_\pi R(\pi)$$

$$\max_\pi R(\pi) = \max_\pi \sum_x R(\mathbf{x}, \pi(\mathbf{x})) P(\mathbf{x}) = \sum_x P(\mathbf{x}) [\max_{\pi(\mathbf{x})} R(\mathbf{x}, \pi(\mathbf{x}))]$$

- **Optimal strategy:** $\pi^* : X \rightarrow A$

$$\pi^*(\mathbf{x}) = \arg\max_a R(\mathbf{x}, a)$$

RL with immediate rewards

- **We know that** $\pi^*(\mathbf{x}) = \arg\max_a R(\mathbf{x}, a)$
- **Problem:** In the RL framework we do not know $R(\mathbf{x}, a)$
  - The expected reward for performing action *a* at input **x**
- **How to get** $R(\mathbf{x}, a)$ **?**

RL with immediate rewards

- **Problem:** In the RL framework we do not know $R(\mathbf{x}, a)$
  - The expected reward for performing action *a* at input **x**
- **Solution:**
  - For each input **x** try different actions *a*
  - Estimate $R(\mathbf{x}, a)$ using the average of observed rewards

$$\tilde{R}(\mathbf{x}, a) = \frac{1}{N_{x,a}} \sum_{i=1}^{N_{x,a}} r_i^{x,a}$$

  - Action choice $\pi(\mathbf{x}) = \arg\max_a \tilde{R}(\mathbf{x}, a)$
  - Accuracy of the estimate: statistics (Hoeffding's bound)

$$P\left(\left|\tilde{R}(\mathbf{x}, a) - R(\mathbf{x}, a)\right| \geq \varepsilon\right) \leq \exp\left[-\frac{2\varepsilon^2 N_{x,a}}{(r_{\max} - r_{\min})^2}\right] \leq \delta$$

  - Number of samples: $N_{x,a} \geq \frac{(r_{\max} - r_{\min})^2}{2\varepsilon^2} \ln \frac{1}{\delta}$

RL with immediate rewards

- **On-line (stochastic approximation)**
  - An alternative way to estimate $R(\mathbf{x}, a)$
- **Idea:**
  - choose action a for input $\mathbf{x}$ and observe a reward $r^{x,a}$
  - Update an estimate

$$\tilde{R}(\mathbf{x}, a) \leftarrow (1-\alpha)\tilde{R}(\mathbf{x}, a) + \alpha\, r^{x,a}$$

α - a learning rate

- **Convergence property:** The approximation converges in the limit for an appropriate learning rate schedule.
- Assume: $\alpha(n(x,a))$ - is a learning rate for nth trial of (x,a) pair
- Then the converge is assured if:

1. $\sum_{i=1}^{\infty} \alpha(i) = \infty$
2. $\sum_{i=1}^{\infty} \alpha(i)^2 < \infty$

Exploration vs. Exploitation

- In the RL framework
  - the (learner) actively interacts with the environment.
  - At any point in time it has an estimate of $\tilde{R}(\mathbf{x}, a)$ for any input action pair
- **Dilemma:**
  - Should the learner use the current best choice of action (exploitation)

$$\hat{\pi}(\mathbf{x}) = \arg\max_{a \in A} \tilde{R}(\mathbf{x}, a)$$

  - Or choose other action a and further improve its estimate (exploration)

- **Different exploration/exploitation strategies exist**

Exploration vs. Exploitation

- **Uniform exploration:** Exploration parameter $0 \leq \varepsilon \leq 1$
  - Choose the "current" best choice with probability $1-\varepsilon$

$$\hat{\pi}(\mathbf{x}) = \arg\max_{a \in A} \tilde{R}(\mathbf{x}, a)$$

  - All other choices are selected with a uniform probability $\frac{\varepsilon}{|A|-1}$
- **Boltzman exploration**
  - The action is chosen randomly but proportionally to its current expected reward estimate

$$p(a \mid \mathbf{x}) = \frac{\exp\left[\tilde{R}(x,a)/T\right]}{\sum_{a' \in A} \exp\left[\tilde{R}(x,a')/T\right]}$$

T – is temperature parameter. **What does it do?**