CS 1675 Introduction to Machine Learning
Lecture 23

Reinforcement learning II

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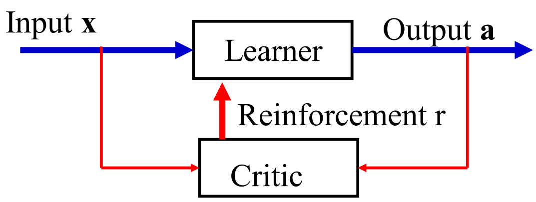
Reinforcement learning

- **We want to learn a control policy:** $\pi : X \rightarrow A$
- We see examples of **x** (but outputs a are not given)
- Instead of a we get a feedback r (reinforcement, reward) from a **critic** quantifying how good the selected output was



- The reinforcements may not be deterministic
- **Goal:** find $\pi : X \rightarrow A$ with the best expected reinforcements

Gambling example

- **Game:** 3 different biased coins are tossed
  - The coin to be tossed is selected randomly from the three options and I always see which coin I am going to play next
  - I make bets on head or tail and I always wage $1
  - If I win I get $1, otherwise I lose my bet
- **RL model:**
  - **Input:** X – a coin chosen for the next toss,
  - **Action:** A – choice of head or tail,
  - **Reinforcements:** {1, -1}
- **A policy** $\pi : X \to A$ **Example:** $\pi :$

$\longrightarrow$ head
$\longrightarrow$ tail
$\longrightarrow$ head

Gambling example

- **RL model:**
  - **Input:** X – a coin chosen for the next toss,
  - **Action:** A – choice of head or tail,
  - **Reinforcements:** {1, -1}
  - **A policy** $\pi : X \to A$
- **Learning goal: find** $\pi^* : X \to A$ **maximizing future expected profits**

$\pi^* :$

$\longrightarrow$?
$\longrightarrow$?
$\longrightarrow$?

$$E(\sum_{t=0}^{T} \gamma^t r_t) \qquad 0 \le \gamma < 1$$

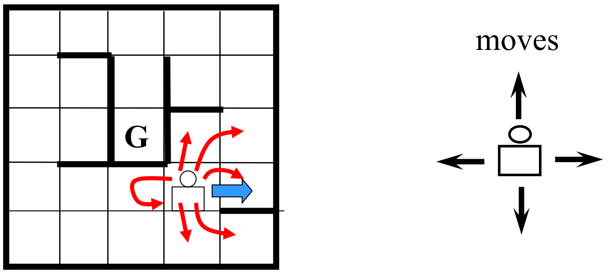
a discount factor = present value of money

RL learning: objective functions

- **Objective:**
  Find a policy $\pi^* : X \rightarrow A$
  That maximizes some combination of future reinforcements (rewards) received over time
- **Valuation models (quantify how good the mapping is):**
  - **Finite horizon models**
    $E(\sum_{t=0}^{T} r_t)$ Time horizon: $T > 0$
    $E(\sum_{t=0}^{T} \gamma^t r_t)$ Discount factor: $0 \leq \gamma < 1$
  - **Infinite horizon discounted model**
    $E(\sum_{t=0}^{\infty} \gamma^t r_t)$ Discount factor: $0 \leq \gamma < 1$
  - **Average reward** $\lim_{T \to \infty} \frac{1}{T} E(\sum_{t=0}^{T} r_t)$

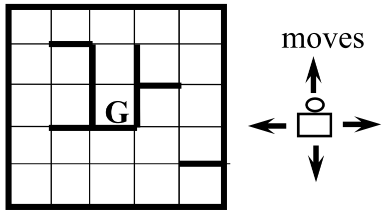
Agent navigation example

- **Agent navigation in the Maze**:
  - 4 moves in compass directions
  - Effects of moves are stochastic – we may wind up in other than intended location with a non-zero probability
  - **Objective:** learn how to reach the goal state in the shortest expected time



Agent navigation example

- **The RL model:**
  - **Input:** X – position of an agent
  - **Output:** A –a move
  - **Reinforcements:** R
    - -1 for each move
    - +100 for reaching the goal
  - **A policy:** $\pi : X \rightarrow A$ **Example:** $\pi :$



Position 1	→	***right***
Position 2	→	***right***
…		
Position 20	→	***left***

- **Goal: find the policy maximizing future expected rewards**

$$E(\sum_{t=0}^{\infty} \gamma^t r_t) \qquad 0 \le \gamma < 1$$

RL with immediate rewards

- **Expected reward**

$$E(\sum_{t=0}^{\infty} \gamma^t r_t) \qquad 0 \le \gamma < 1$$

- **Immediate reward case:**
  - Reward for the choice becomes available immediately
  - Our action does not affect the environment and thus future rewards

$$E(\sum_{t=0}^{\infty} \gamma^t r_t) = E(r_0) + E(\gamma r_1) + E(\gamma^2 r_2) + \ldots$$

$r_0, r_1, r_2 \cdots$ Rewards for every step of the game

  - Expected one step reward for input **x (coin to play next)** and the choice a : $R(\mathbf{x}, a)$

RL with immediate rewards

- **Expected reward**

$$E(\sum_{t=0}^{\infty}\gamma^t r_t) = E(r_0) + E(\gamma r_1) + E(\gamma^2 r_2) + \dots$$

- **Optimal strategy:**

$$\pi^* : X \rightarrow A$$

$$\pi^*(\mathbf{x}) = \arg\max_a R(\mathbf{x}, a)$$

where $R(\mathbf{x}, a) = \sum_j r(\omega_j \mid a, \mathbf{x}) P(\omega_j \mid \mathbf{x}, a)$

is an Expected reward for the input x and choice a

- ω_j - outcome of action for x that determines the reward (e.g. an outcome of the coin toss)

RL with immediate rewards

The optimal choice assumes we know the expected reward $R(\mathbf{x}, a)$

- **Then:** $\pi^*(\mathbf{x}) = \arg\max_a R(\mathbf{x}, a)$

Caveats

- **We do not know the expected reward** $R(\mathbf{x}, a)$
  - We need to estimate it using $\tilde{R}(\mathbf{x}, a)$ from interaction
- **We cannot determine the optimal policy if the estimate of the expected reward is not good**
  - We need to try also actions that look suboptimal wrt the current estimates of $\tilde{R}(\mathbf{x}, a)$

Estimating R(x,*a*)

- **Solution 1:**
  - For each input **x** try different actions *a*
  - Estimate $R(\mathbf{x},a)$ using the average of observed rewards

$$\widetilde{R}(\mathbf{x},a) = \frac{1}{N_{x,a}} \sum_{i=1}^{N_{x,a}} r_i^{x,a}$$

- **Solution 2: online approximation**
- Updates an estimate after performing action *a* in x and observing the reward $r^{x,a}$

$$\widetilde{R}(\mathbf{x},a)^{(i)} \leftarrow (1-\alpha(i))\widetilde{R}(\mathbf{x},a)^{(i-1)} + \alpha(i)\, r_i^{x,a}$$

$\alpha(i)$ - a learning rate

Exploration vs. Exploitation

- **Uniform exploration:** Exploration parameter $0 \le \varepsilon \le 1$
  - Choose the "current" best choice with probability $1-\varepsilon$

$$\hat{\pi}(\mathbf{x}) = \arg\max_{a \in A} \widetilde{R}(\mathbf{x},a)$$

  - All other choices are selected with a uniform probability $\frac{\varepsilon}{|A|-1}$
- **Boltzman exploration**
  - The action is chosen randomly but proportionally to its current expected reward estimate

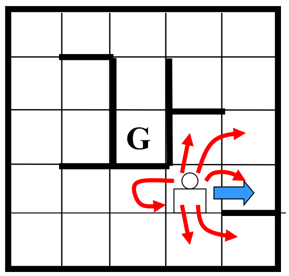
$$p(a \mid \mathbf{x}) = \frac{\exp\left[\widetilde{R}(x,a)/T\right]}{\sum_{a' \in A} \exp\left[\widetilde{R}(x,a')/T\right]}$$

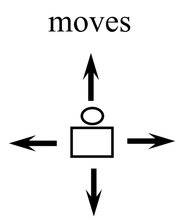
T is tuned gradually from high to low values

RL with delayed rewards

A more general reinforcement learning model

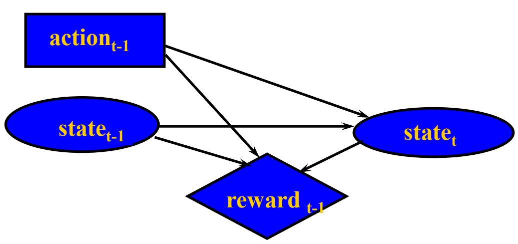
- **Agent navigation in the Maze**:
  - 4 moves in compass directions
  - Effects of moves are stochastic – we may wind up in other than intended location with non-zero probability
  - **Objective:** reach the goal state in the shortest time



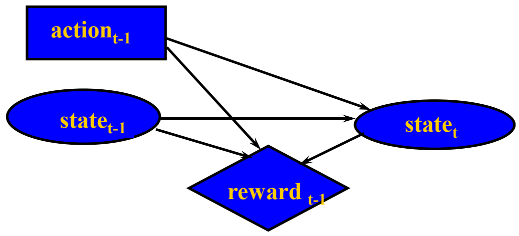


Learning with delayed rewards

- Actions, in addition to immediate rewards affect the next state of the environment and thus indirectly also future rewards
- We need a model to represent environment changes
- The model we use is called **Markov decision process (MDP)**
  - Frequently used in AI, OR, control theory
  - **Markov assumption**: next state depends on the previous state and action, and not states (actions) in the past



Markov decision process



Formal definition: 4-tuple (S, A, T, R)

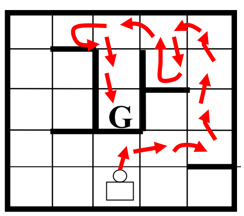
• **A set of states** S (X)	locations of a robot
• **A set of actions** A	move actions
• **Transition model** $S \times A \times S \rightarrow [0,1]$	where can I get with different moves
• **Reward model** $S \times A \times S \rightarrow \Re$	reward/cost for a transition

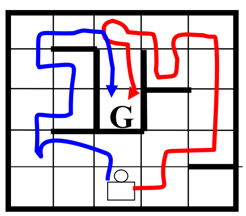
MDP problem

- We want to find the best policy $\pi^* : S \rightarrow A$
- **Value function** (V) for a policy, quantifies the goodness of a policy through, e.g. infinite horizon, discounted model

$$E(\sum_{t=0}^{\infty} \gamma^t r_t)$$

It: 1. combines future rewards over a trajectory
2. combines rewards for multiple trajectories (through expectation-based measures)





Value of a policy for MDP

- Assume a fixed policy $\pi : S \rightarrow A$
- How to compute the value of a policy under infinite horizon discounted model?

A fixed point equation:

$$V^{\pi}(s) = \underbrace{R(s,\pi(s))}_{\text{expected one step reward for the first action}} + \underbrace{\gamma \sum_{s' \in S} P(s' \mid s, \pi(s)) V^{\pi}(s')}_{\text{expected discounted reward for following the policy for the rest of the steps}}$$

expected one step reward for the first action

expected discounted reward for following the policy for the rest of the steps

$$\mathbf{v} = \mathbf{r} + \mathbf{U}\mathbf{v} \quad \Longrightarrow \quad \mathbf{v} = (\mathbf{I} - \mathbf{U})^{-1}\mathbf{r}$$

– For a finite state space– we get a set of linear equations

Optimal policy

- **The value of the optimal policy**

$$V^{*}(s) = \max_{a \in A}\left[\underbrace{R(s,a)}_{} + \underbrace{\gamma \sum_{s' \in S} P(s' | s,a) V^{*}(s')}_{} \right]$$

expected one step reward for the first action

expected discounted reward for following the opt. policy for the rest of the steps

Value function mapping form:

$$V^{*}(s) = (HV^{*})(s)$$

- **The optimal policy:** $\pi^{*} : S \rightarrow A$

$$\pi^{*}(s) = \arg\max_{a \in A}\left[R(s,a) + \gamma \sum_{s' \in S} P(s' | s,a) V^{*}(s') \right]$$

Computing optimal policy

Dynamic programming: **Value iteration:**
- computes the optimal value function first then the policy
- iterative approximation
- converges to the optimal value function

Value iteration (ε **)**

initialize $\mathbf{V}$;; V is vector of values for all states

repeat

set $\mathbf{V'} \leftarrow \mathbf{V}$

set $\mathbf{V} \leftarrow \mathbf{HV}$

until $\|\mathbf{V'} - \mathbf{V}\|_\infty \leq \varepsilon$

output $\pi^*(s) = \arg\max_{a \in A} \left[R(s,a) + \gamma \sum_{s' \in S} P(s' \mid s,a) V(s') \right]$

Reinforcement learning of optimal policies

- **In the RL framework we do not know the MDP model !!!**
- **Goal:** learn the optimal policy

$$\pi^* : S \rightarrow A$$

- **Two basic approaches:**
  - **Model based learning**
    - Learn the MDP model (probabilities, rewards) first
    - Solve the MDP afterwards
  - **Model-free learning**
    - Learn how to act directly
    - No need to learn the parameters of the MDP
  - A number of clones of the two in the literature

Model-based learning

- We need to learn **transition probabilities** and **rewards**
- **Learning of probabilities**
  - ML parameter estimates
  - Use counts $\tilde{P}(s'|s,a) = \frac{N_{s,a,s'}}{N_{s,a}}$ $\qquad N_{s,a} = \sum_{s' \in S} N_{s,a,s'}$
- **Learning rewards**
  - Similar to learning with immediate rewards

$$\tilde{R}(s,a) = \frac{1}{N_{s,a}} \sum_{i=1}^{N_{s,a}} r_i^{s,a}$$ or **the online solution**

- **Problem: changes in the probabilities and reward estimates would require us to solve an MDP from scratch !** **(after every action and reward seen)**

Model free learning

- **Motivation:** value function update (value iteration):

$$V^*(s) \leftarrow \max_{a \in A} \left[R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^*(s') \right]$$

- Let

$$Q(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^*(s')$$

- Then $\quad V^*(s) \leftarrow \max_{a \in A} Q(s,a)$

- Note that the update can be defined purely in terms of Q-functions

$$Q(s,a) \leftarrow R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) \max_{a'} Q(s',a')$$

Q-learning

- **Q-learning** uses the Q-value update idea
  - **But** relies on a stochastic (on-line, sample by sample) update

$$Q(s,a) \leftarrow R(s,a) + \gamma \sum_{s' \in S} P(s' | s,a) \max_{a'} Q(s',a')$$

is replaced with

$$\hat{Q}(s,a) \leftarrow (1-\alpha)\hat{Q}(s,a) + \alpha\left(r(s,a) + \gamma \max_{a'} \hat{Q}(s',a')\right)$$

$r(s,a)$ - reward received from the environment after performing an action *a* in state *s*

s' - new state reached after action *a*

α - learning rate, a function of $N_{s,a}$ - a number of times *a* executed at *s*

Q-learning

The on-line update rule is applied repeatedly during the direct interaction with an environment

Q-learning
initialize *Q(s,a) =0* for all *s,a* pairs
observe current state *s*
repeat
  select action *a* ; use some exploration/exploitation schedule
  receive reward *r*
  observe next state *s'*
  update $Q(s,a) \leftarrow (1-\alpha)Q(s,a) + \alpha\left(r + \gamma \max_{a'} Q(s',a')\right)$
  set *s* to *s'*
end repeat

Q-learning convergence

The **Q-learning is guaranteed to converge** to the optimal Q-values under the following conditions:

- Every state is visited and every action in that state is tried infinite number of times
  - This is assured via exploration/exploitation schedule
- The sequence of learning rates for each *Q(s,a)* satisfies:

1. $\sum_{i=1}^{\infty} \alpha(i) = \infty$ 2. $\sum_{i=1}^{\infty} \alpha(i)^2 < \infty$

$\alpha(n(s,a))$ - is the learning rate for the *n*th trial of (*s*,*a*)

RL with delayed rewards

The optimal choice $\pi^*(\mathbf{s}) = \arg\max_a Q(s,a)$

- **much like what we had for the immediate rewards**

$$\pi^*(\mathbf{x}) = \arg\max_a R(\mathbf{x},a)$$

RL Learning

- **Instead of exact values of** $Q(\mathbf{s},a)$ **we use** $\hat{Q}(\mathbf{s},a)$

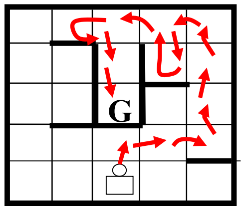
$$\hat{Q}(s,a) \leftarrow (1-\alpha)\hat{Q}(s,a) + \alpha\left(r(s,a) + \gamma \max_{a'} \hat{Q}(s',a')\right)$$

- **Since we have only estimates of** $\hat{Q}(\mathbf{s},a)$
  - We need to try also actions that look suboptimal wrt the current estimates
  - Exploration/exploitation strategies
    - **Uniform exploration**
    - **Boltzman exploration**

Q-learning speed-ups

- The basic Q-learning rule updates may propagate distant (delayed) rewards very slowly

Example:



- **Goal:** a high reward state
- To make the correct decision we need all Q-values for the current position to be good
- **Problem:**
  - in each run we back-propagate values only 'one-step' back. It takes multiple trials to back-propagate values multiple steps.

Q-learning speed-ups

- **Remedy:** Backup values for a larger number of steps

Rewards from applying the policy

$$q_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \ldots = \sum_{i=0}^{\infty} \gamma^i r_{t+i}$$

We can substitute (immediate rewards with n-step rewards):

$$q_t^n = \sum_{i=0}^{n} \gamma^i r_{t+i} + \gamma^{n+1} \max_{a'} Q_{t+n}(s', a')$$

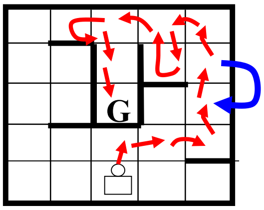
Postpone the update for *n* steps and update with a longer trajectory rewards

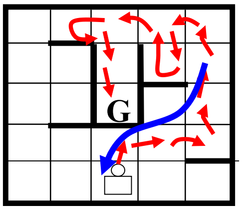
$$Q_{t+n+1}(s,a) \leftarrow Q_{t+n}(s,a) + \alpha\left(q_t^n - Q_{t+n}(s,a)\right)$$

Problems: - larger variance
- exploration/exploitation switching
- wait n steps to update

Q-learning speed-ups

- One step vs. n-step backup



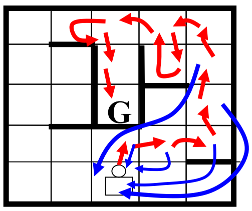


Problems with n-step backups:

- larger variance
- exploration/exploitation switching
- wait n steps to update

Q-learning speed-ups

- **Temporal difference (TD) method**
  - Remedy of the wait n-steps problem
  - Partial back-up after every simulation step
    - Similar idea: weather forecast adjustment



Different versions of this idea has been implemented

RL successes

- Reinforcement learning is relatively simple
  - On-line techniques can track non-stationary environments and adapt to its changes

- **Successful applications**:
  - **AlphaGo**
  - TD Gammon – learned to play backgammon on the championship level
  - Elevator control
  - Dynamic channel allocation in mobile telephony
  - Robot navigation in the environment

Next lecture (Tuesday)

- Tools to support machine learning