Decision making in the presence of uncertainty

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Decision-making in the presence of uncertainty

- Computing the probability of some event may not be our ultimate goal
- Instead we are often interested in making decisions about our future actions so that we satisfy some goals
- Example: medicine
  - Diagnosis is typically only the first step
  - The ultimate goal is to manage the patient in the best possible way. Typically many options available:
    - Surgery, medication, collect the new info (lab test)
    - There is an uncertainty in the outcomes of these procedures: patient can be improve, get worse or even die as a result of different management choices.
Decision-making in the presence of uncertainty

Main issues:
- How to model the decision process with uncertain outcomes in the computer?
- How to make decisions about actions in the presence of uncertainty?

The field of decision-making studies ways of making decisions in the presence of uncertainty.

Decision making example.
Assume we want to invest $100 for 6 months
- We have 4 choices:
  1. Invest in Stock 1
  2. Invest in Stock 2
  3. Put money in bank
  4. Keep money at home

Stock 1 value can go up or down:
- Up: with probability 0.6
- Down: with probability 0.4
Decision making example.

Assume we want to invest $100 for 6 months

- **We have 4 choices:**
  1. Invest in Stock 1
  2. Invest in Stock 2
  3. Put money in bank
  4. Keep money at home

Stock 1 value can go **up** or **down**:
- **Up**: with probability 0.6
- **Down**: with probability 0.4

Monetary outcomes for up and down states

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Decision making example.

**Investing of $100 for 6 months**

Monetary outcomes for different states
Decision making example.

We need to make a choice whether to invest in Stock 1 or 2, put money into bank or keep them at home. But how?

<table>
<thead>
<tr>
<th>Stock 1</th>
<th>Stock 2</th>
<th>Bank</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>101</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>110</td>
</tr>
</tbody>
</table>

Monetary outcomes for different scenarios:

- (up) 110
- (down) 90
- (up) 140
- (down) 80

Home: 100

Decision making example.

Assume a simplified problem with the Bank and Home choices only.

The result is guaranteed – the outcome is deterministic.

<table>
<thead>
<tr>
<th>Bank</th>
<th>Home</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

What is the rational choice assuming our goal is to make money?
Decision making. Deterministic outcome.

Assume a simplified problem with the Bank and Home choices only.
These choices are deterministic.

Our goal is to make money. What is the rational choice?
Answer: Put money into the bank. The choice is always strictly better in terms of the outcome.
But what to do if we have uncertain outcomes?

Decision making. Stochastic outcome

- How to quantify the goodness of the stochastic outcome?
We want to compare it to deterministic and other stochastic outcomes.
Decision making. Stochastic outcome

- How to quantify the goodness of the stochastic outcome?
  We want to compare it to deterministic and other stochastic outcomes.

Idea: Use the expected value of the outcome

Expected value

- Let $X$ be a random variable representing the monetary outcome with a discrete set of values $\Omega_X$.
- Expected value of $X$ is:
  $$E(X) = \sum_{x \in \Omega_X} xP(X = x)$$

**Intuition:** Expected value summarizes all stochastic outcomes into a single quantity.

- Example:

  - What is the expected value of the outcome of Stock 1 option?
Expected value

- Let $X$ be a random variable representing the monetary outcome with a discrete set of values $\Omega_X$.
- **Expected value** of $X$ is:
  \[ E(X) = \sum_{x \in \Omega_X} x P(X = x) \]
- **Expected value** summarizes all stochastic outcomes into a single quantity

- **Example:**
  \[
  \text{Expected value for the outcome of the Stock 1 option is: } \\
  0.6 \times 110 + 0.4 \times 90 = 66 + 36 = 102
  \]

Expected values

**Investing $100 for 6 months**

- **Stock 1**
  - (up) $110$ with $0.6$ probability
  - (down) $90$ with $0.4$ probability

- **Stock 2**
  - (up) $140$ with $0.4$ probability
  - (down) $80$ with $0.6$ probability

- **Bank**
  - $101$ with $1.0$ probability

- **Home**
  - $100$ with $1.0$ probability

\[
\text{Expected value for Stock 1: } 0.6 \times 110 + 0.4 \times 90 = 66 + 36 = 102
\]
Investing $100 for 6 months

Expected values

Stock 1
- 102
- 0.6 (up) ➔ 110
- 0.4 (down) ➔ 90

Stock 2
- 104
- 0.4 (up) ➔ 140
- 0.6 (down) ➔ 80

Bank
- 1.0 ➔ 101

Home
- 1.0 ➔ 100

\[0.6 \times 110 + 0.4 \times 90 = 66 + 36 = 102\]

\[0.4 \times 140 + 0.6 \times 80 = 56 + 48 = 104\]

Expected values

Investing $100 for 6 months

Expected values

Investing $100 for 6 months

Expected values

Investing $100 for 6 months

Expected values

Investing $100 for 6 months

Expected values

Investing $100 for 6 months

Expected values
Expected values

Investing $100 for 6 months

- **Stock 1**
  - (up) 102
  - (down) 110

- **Stock 2**
  - (up) 104
  - (down) 140

- **Bank**
  - 101

- **Home**
  - 101

**Calculations**

- **Stock 1**
  - $0.6 \times 110 + 0.4 \times 90 = 66 + 36 = 102$

- **Stock 2**
  - $0.4 \times 140 + 0.6 \times 80 = 56 + 48 = 104$

- **Home**
  - $1.0 \times 101 = 101$

---

Expected values

Investing $100 for 6 months

- **Stock 1**
  - (up) 102
  - (down) 110

- **Stock 2**
  - (up) 104
  - (down) 140

- **Bank**
  - 101

- **Home**
  - 101

**Calculations**

- **Stock 1**
  - $0.6 \times 110 + 0.4 \times 90 = 66 + 36 = 102$

- **Stock 2**
  - $0.4 \times 140 + 0.6 \times 80 = 56 + 48 = 104$

- **Home**
  - $1.0 \times 101 = 101$

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Expected values

Investing $100 for 6 months

Stock 1

- 102
- 0.6 (up)
- 0.4 (down)
- 110
- 90

Stock 2

- 104
- 0.4 (up)
- 0.6 (down)
- 140
- 80

Bank

- 101
- 1.0
- 101

Home

- 100
- 1.0
- 100

0.6 × 110 + 0.4 × 90 = 66 + 36 = 102

0.4 × 140 + 0.6 × 80 = 56 + 48 = 104

1.0 × 101 = 101

1.0 × 100 = 100

Selection based on expected values

The optimal action is the option that maximizes the expected outcome:
Relation to the game search

- **Game search**: minimax algorithm
  - considers the rational opponent and its best move
- **Decision making**: maximizes the expectation
  - play against the nature – a stochastic non-malicious “opponent”

(Stochastic) Decision tree

- **Decision tree**:
  - □ decision node
  - ● chance node
  - ▪ outcome (value) node

---

```
Stock 1
102   0.6  (up) 110
     0.4  (down) 90

Stock 2
104   0.4  (up) 140
     0.6  (down) 80

Bank
101   1.0  ▪ 101

Home
100   1.0  ▪ 100
```
Sequential (multi-step) problems

The decision tree can be built to capture multi-step decision problems:

- Choose an action
- Observe the stochastic outcome
- And repeat

How to make decisions for multi-step problems?

- Start from the leaves of the decision tree (outcome nodes)
- Compute expectations at chance nodes
- Maximize at the decision nodes

Algorithm is sometimes called **expectimax**

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Multi-step problem example

Assume:

- Two investment periods
- Two actions: stock and bank

```
<table>
<thead>
<tr>
<th></th>
<th>Stock (up)</th>
<th>Stock (down)</th>
<th>Bank (up)</th>
<th>Bank (down)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period 1</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Period 2</td>
<td>0.4</td>
<td>0.6</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>
```

---
Multi-step problem example

Assume:
- Two investment periods
- Two actions: stock and bank
Multi-step problem example

Assume:
- Two investment periods
- Two actions: stock and bank

```
Stock  Bank
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>(up)</td>
<td>(down)</td>
</tr>
</tbody>
</table>

150 125 95 90 110 105

Stock  Bank
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>(down)</td>
<td>(down)</td>
</tr>
</tbody>
</table>

150 95 90 110 105
```

```
Stock  Bank
<p>| | |</p>
<table>
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</table>

150 125 95 90 110 105

Stock  Bank
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</tr>
<tr>
<td>(up)</td>
<td>(down)</td>
</tr>
</tbody>
</table>

150 95 90 110 105
```
Multi-step problem example

Assume:

- Two investment periods
- Two actions: stock and bank


- Notice that the probability of stock going up and down in the 2nd step is independent of the 1st step (=0.5)
Conditioning in the decision tree

- But this may not hold in general. In decision trees:
  - Later outcomes can be conditioned on the earlier stochastic outcomes and actions

**Example:** stock movement probabilities. Assume:

- \( P(1^{st}=up)=0.4 \)
- \( P(2^{nd}=up|1^{st}=up)=0.4 \)
- \( P(2^{nd}=up|1^{st}=down)=0.5 \)

---

**Multi-step problems. Conditioning.**

**Tree Structure:** every observed stochastic outcome = 1 branch

- \( P(1^{st}=up)=0.4 \)
- \( P(2^{nd}=up|1^{st}=up)=0.4 \)
- \( P(2^{nd}=up|1^{st}=down)=0.5 \)
Trajectory payoffs

- **Outcome values at leaf nodes (e.g. monetary values)**
  - Rewards and costs for the path trajectory

**Example:** stock fees and gains. **Assume:**
- Fee per period: $5 paid at the beginning
- Gain for up: 15%, loss for down 10%

```
Stock
1000-5
Bank
```

### Constructing a decision tree

- **The decision tree is rarely given to you directly.**
  - Part of the problem is to construct the tree.

**Example:** stocks, bonds, bank for k periods

**Stock:**
- Probability of stocks going up in the first period: 0.3
- Probability of stocks going up in subsequent periods:
  - \( P(k\text{th step}=\text{Up}| (k-1)\text{th step}=\text{Up})=0.4 \)
  - \( P(k\text{th step}=\text{Up}| (k-1)\text{th step}=\text{Down})=0.5 \)
- Return if stock goes up: 15% if down: 10%
- Fixed fee per investment period: $5

**Bonds:**
- Probability of value up: 0.5, down: 0.5
- Return if bond value is going up: 7%, if down: 3%
- Fee per investment period: $2

**Bank:**
- Guaranteed return of 3% per period, no fee
**Information-gathering actions**

- Many actions and their outcomes irreversibly change the world
- **Information-gathering (exploratory) actions:**
  - make an inquiry about the world
  - **Key benefit:** reduction in the uncertainty
- **Example: medicine**
  - Assume a patient is admitted to the hospital with some set of initial complaints
  - We are uncertain about the underlying problem and consider a surgery, or a medication to treat them
  - But there are often lab tests or observations that can help us to determine more closely the disease the patient suffers from
  - **Goal of lab tests:** Reduce the uncertainty of outcomes of treatments so that better treatment option can be chosen

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**Decision-making with exploratory actions**

**In decision trees:**

- **Exploratory actions** can be represented and reasoned about the same way as other actions.

How do we capture the effect of exploratory actions in the decision tree model?

- Information obtained through exploratory actions may affect the probabilities of later outcomes
  - Recall that the probabilities on later outcomes can be conditioned on past observed outcomes and past actions
  - Sequence of past actions and outcomes is “remembered” within the decision tree branch
Oil wildcatter problem.

An oil wildcatter has to make a decision of whether to drill or not to drill on a specific site

- **Chance of hitting an oil deposit:**
  - Oil: 40% \( P(Oil = T) = 0.4 \)
  - No-oil: 60% \( P(Oil = F) = 0.6 \)

- **Cost of drilling:** 70K

- **Payoffs:**
  - Oil: 220K
  - No-oil: 0 K

\[
\begin{align*}
\text{Drill} & : 0.4 \rightarrow 220-70=150 \\
\text{No-drill} & : 0.6 \rightarrow -70
\end{align*}
\]
Oil wildcatter problem

- Assume that in addition to the drill/no-drill choices we have an option to run the **seismic resonance test**
- **Seismic resonance test results:**
  - Closed pattern (more likely when the hole holds the oil)
  - Diffuse pattern (more likely when empty)

<table>
<thead>
<tr>
<th>Seismic resonance test pattern</th>
<th>closed</th>
<th>diffuse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oil True</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>Oil False</td>
<td>0.3</td>
<td>0.7</td>
</tr>
</tbody>
</table>

- **Test cost:** 10K

Oil wildcatter problem.

- **Decision tree**

```plaintext
Test 0.5

0.5

Drill 18

0.4

220-70=150

0.6

(no-oil) -70

No-drill 0

1.0

220-70=140

0.64

(oil)

0-70-10=-80

0-10=-10

18

Drill 0

0.36

(no-oil)

0-70-10=-80

0-10=-10

0.16

(oil)

220-70-10=140

0.84

(no-oil)

0-70-10=-80

0-10=-10

1.0

(oil)
```
Oil wildcatter problem.

- Alternative model

Oil wildcatter problem.

- Decision tree probabilities

\[ P(Oil \mid Test = closed) = \frac{P(Test = closed \mid Oil = T)P(Oil = T)}{P(Test = closed)} = \frac{0.8 \times 0.4}{0.8 \times 0.4 + 0.6 \times 0.2} = 0.64 \]

\[ P(Oil = F \mid Test = closed) = \frac{P(Test = closed \mid Oil = F)P(Oil = F)}{P(T = closed)} = 0.36 \]

\[ P(Test = closed) = P(Test = closed \mid Oil = F)P(Oil = F) + P(Test = closed \mid Oil = T)P(Oil = T) = 0.5 \]
Oil wildcatter problem.

- Decision tree probabilities

Drill

\[ P(\text{Test}) \]

\[
P(\text{Test} = \text{closed}) = P(\text{Test} = \text{closed} | \text{Oil} = F)P(\text{Oil} = F) + P(\text{Test} = \text{closed} | \text{Oil} = T)P(\text{Oil} = T) 
\]

\[
P(\text{Test} = \text{diff}) = P(\text{Test} = \text{diff} | \text{Oil} = F)P(\text{Oil} = F) + P(\text{Test} = \text{diff} | \text{Oil} = T)P(\text{Oil} = T) 
\]
Oil wildcatter problem.

- Decision tree

The presence of the test and its result affected our decision:
if test = closed then drill
if test = diffuse then do not drill

Value of information

- When the test makes sense?
  - Only when its result makes the decision maker to change his mind, that is he decides not to drill.

- Value of information:
  - Measure of the goodness of the information from the test
  - Difference between the expected value with and without the test information

- Oil wildcatter example:
  - Expected value without the test = 18
  - Expected value with the test = 25.4
  - Value of information for the seismic test = 7.4