# CS 1571 Introduction to AI
# Lecture 8

# Constraint satisfaction search. Combinatorial optimization search.

**Milos Hauskrecht**
milos@cs.pitt.edu
5329 Sennott Square



## Constraint satisfaction problem (CSP)

**Objective:**
- **Find a configuration** satisfying goal conditions

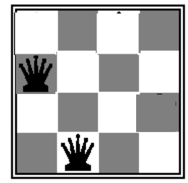
- **Constraint satisfaction problem (CSP)** is **a configuration search problem** where:
  - A **state** is defined by **a set of variables and their values**
  - **Goal condition** is represented by **a set constraints on possible variable values**

# CSP example: N-queens

**Goal:** n queens placed in non-attacking positions on the board

**Variables:**

- Represent queens, one for each column:
  - $Q_1, Q_2, Q_3, Q_4$
- Values:
  - Row placement of each queen on the board {1, 2, 3, 4}



$Q_1 = 2, Q_2 = 4$

**Constraints:** $Q_i \neq Q_j$ Two queens not in the same row

$|Q_i - Q_j| \neq |i - j|$ Two queens not on the same diagonal



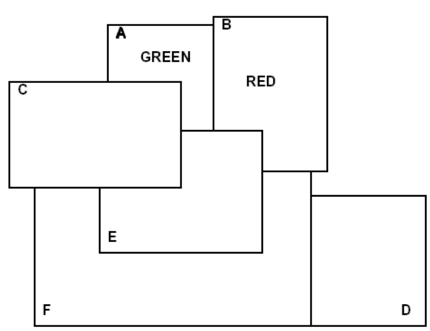
# CSP example: Map coloring

Color a map using k different colors such that no adjacent countries have the same color

**Variables: ?**

- Variable values: ?

**Constraints: ?**



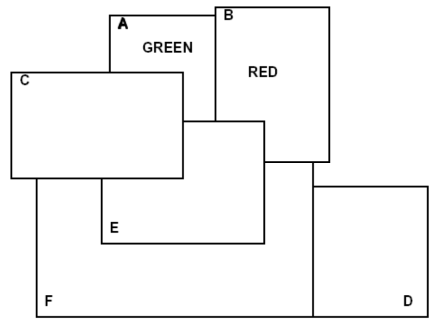
# Map coloring

Color a map using k different colors such that no adjacent countries have the same color

**Variables:**

- Represent countries
  - $A, B, C, D, E$
- Values:
  - K -different colors

  {Red, Blue, Green,..}




**Constraints: ?**



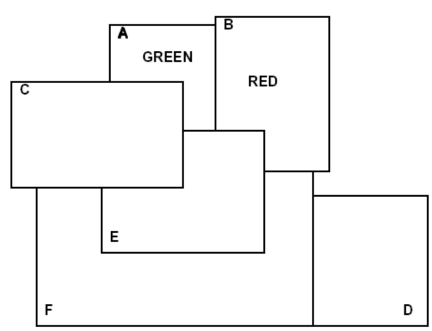
# Map coloring

Color a map using k different colors such that no adjacent countries have the same color

**Variables:**

- Represent countries
  - $A, B, C, D, E$
- Values:
  - K -different colors

  {Red, Blue, Green,..}




**Constraints:** $A \neq B, A \neq C, C \neq E$, etc

An example of a problem with **binary constraints**

# Constraint satisfaction as a search problem

**Formulation of a CSP as a search problem:**

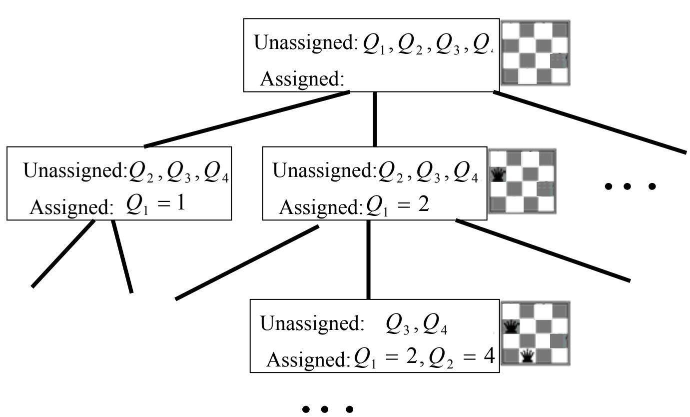
- **States.** Assignment (partial, complete) of values to variables.
- **Initial state.** No variable is assigned a value.
- **Operators.** Assign a value to one of the unassigned variables.
- **Goal condition.** All variables are assigned, no constraints are violated.

- **Constraints** can be **represented**:
  - **Explicitly** by a set of allowable values
  - **Implicitly** by a function that tests for the satisfaction of constraints



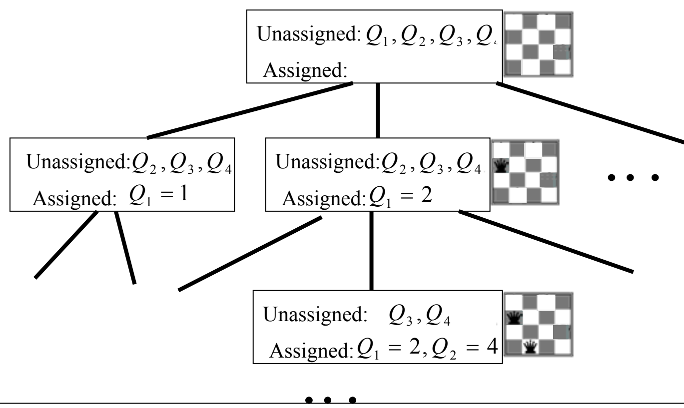
# Solving a CSP through standard search

- **Maximum depth of the tree (m): ?**
- **Depth of the solution (d) : ?**
- **Branching factor (b) : ?**



# Solving a CSP through standard search

- **Maximum depth of the tree:** Number of variables in the CSP
- **Depth of the solution:** Number of variables in the CSP
- **Branching factor:** if we fix the order of variable assignments the branch factor depends on the number of their values

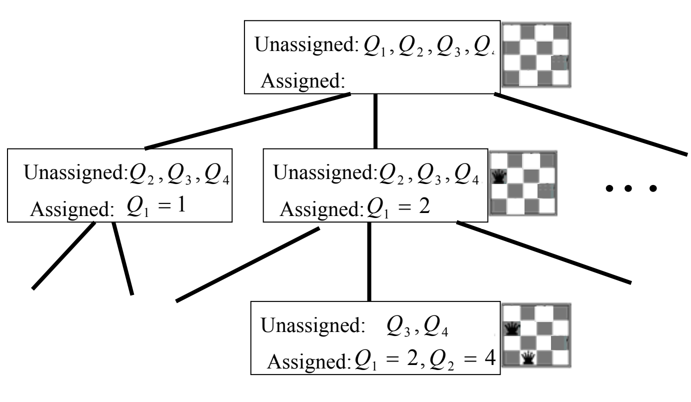





# Solving a CSP through standard search

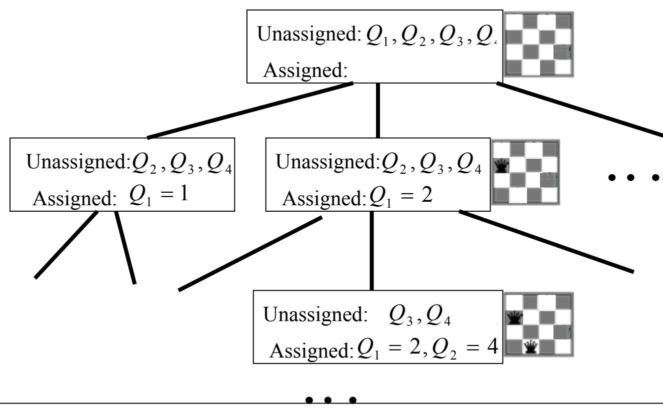
- **What search algorithm to use: ?**

  Depth of the tree = Depth of the solution=number of vars



# Solving a CSP through standard search

- **What search algorithm to use: Depth first search !!!**
  - Since we know the depth of the solution
  - We do not have to keep large number of nodes in queues






# Backtracking

**Depth-first search for CSP** is also referred to as **backtracking**

The violation of constraints needs to be checked for each node, either during its generation or before its expansion

**Consistency of constraints:**

- Current **variable assignments** together **with constraints restrict remaining legal values of unassigned variables**;
- The remaining **legal and illegal values of variables may be inferred** (effect of constraints propagates)

# Constraint propagation

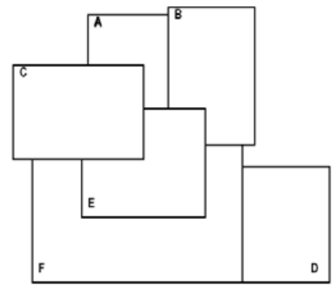
A **state** (more broadly) is defined by a set of variables, their values and a list of legal and illegal assignments for unassigned variables

Legal and illegal assignments can be represented via: **equations** (**list of value assignments**) and **disequations** (**list of invalid assignments**)

**Example:** **map coloring**

Equation $A = \text{Red}$

Disequation $C \neq \text{Red}$




**Constraints + assignments**
can entail new equations and disequations

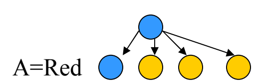
$A = \text{Red} \rightarrow B \neq \text{Red}$

**Constraint propagation:** the process of inferring of new equations and disequations from existing equations and disequations

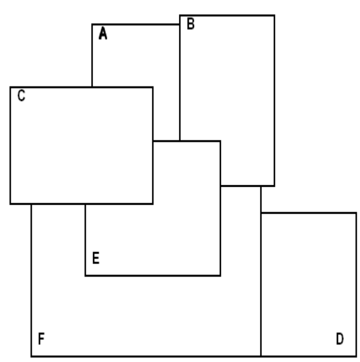


# Constraint propagation

- Assign A=Red

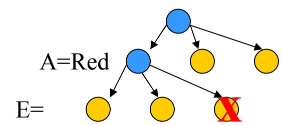



| | Red | Blue | Green |
|---|---|---|---|
| A | ✓ | | |
| B | | | |
| C | | | |
| D | | | |
| E | | | |
| F | | | |



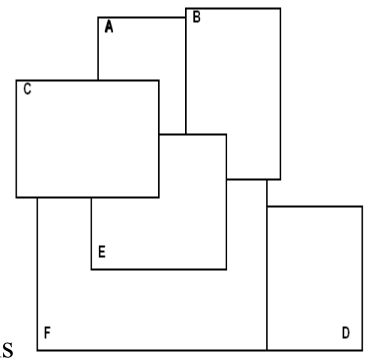

✓ - equations ✗ - disequations

# Constraint propagation




- Assign A=Red

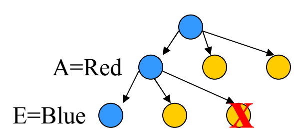
| | Red | Blue | Green |
|---|---|---|---|
| A | ✓ | | |
| B | ✗ | | |
| C | ✗ | | |
| D | | | |
| E | ✗ | | |
| F | | | |




✓ - equations ✗ - disequations

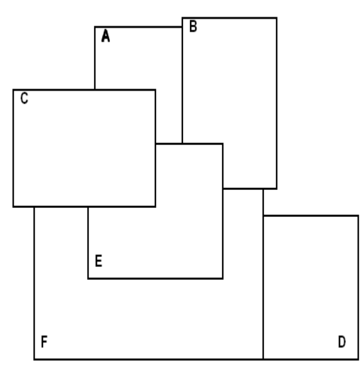


# Constraint propagation




- Assign E=Blue

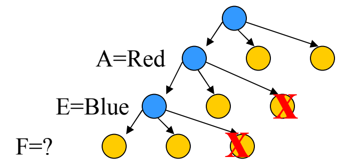
| | Red | Blue | Green |
|---|---|---|---|
| A | ✓ | | |
| B | ✗ | | |
| C | ✗ | | |
| D | | | |
| E | ✗ | ✓ | |
| F | | | |

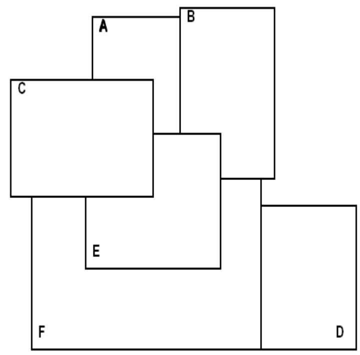


# Constraint propagation

- Assign E=Blue

| | Red | Blue | Green |
|---|---|---|---|
| A | ✓ | ✗ | |
| B | ✗ | ✗ | |
| C | ✗ | ✗ | |
| D | | | |
| E | ✗ | ✓ | |
| F | | ✗ | |



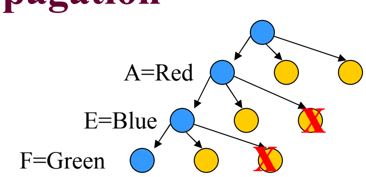



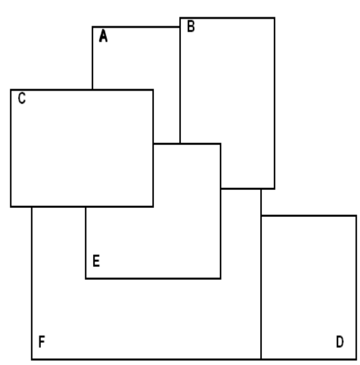



# Constraint propagation

- Assign F=Green

| | Red | Blue | Green |
|---|---|---|---|
| A | ✓ | ✗ | |
| B | ✗ | ✗ | |
| C | ✗ | ✗ | |
| D | | | |
| E | ✗ | ✓ | |
| F | | ✗ | ✓ |



## Constraint propagation

Three known techniques for propagating the effects of past assignments and constraints:

- **Value propagation**
- **Arc consistency**
- **Forward checking**

- **Difference:**
  – Completeness of inferences
  – Time complexity of inferences.



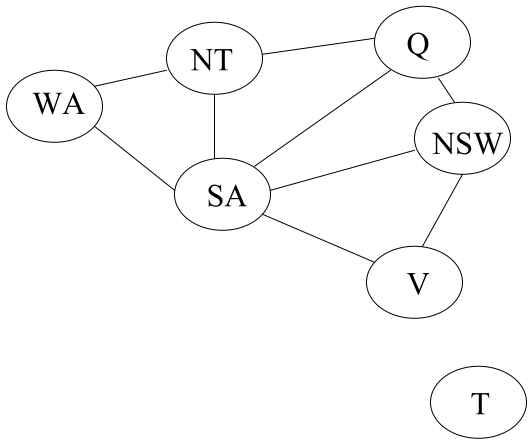
## Constraint propagation

1. Value propagation. **Infers:**
   – **equations** from the set of **equations** defining the partial assignment, **and a constraint**
2. Arc consistency. **Infers:**
   – **disequations** from the set of **equations and disequations** defining the partial assignment, and **a constraint**
   – **equations** through **the exhaustion of alternatives**
3. Forward checking. **Infers:**
   – **disequations** from a set of **equations** defining the partial assignment, and a constraint
   – **Equations** through **the exhaustion of alternatives**

   **Restricted forward checking:**
   – uses only active constraints (active constraint – only one variable unassigned in the constraint)

# Example

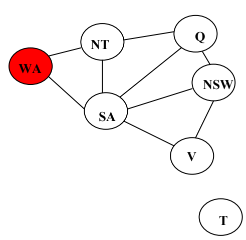
**Map coloring of Australia territories**






# Example: forward checking

**Map coloring**

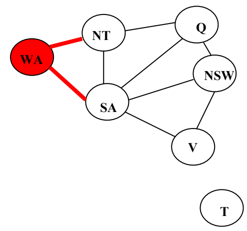



**Set: WA=Red**

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | ? | ? | ? | ? | ? | ? |

# Example: forward checking

**Map coloring**



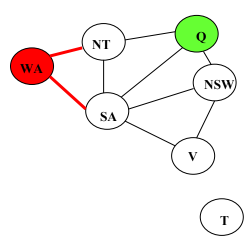

**Set: WA=Red**

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | G B | R G B | R G B | R G B | G B | R G B |



# Example: forward checking

**Map coloring**

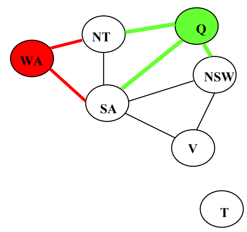



**Set: Q=Green**

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | G B | R G B | R G B | R G B | G B | R G B |
| *Q=Green* | **R** | ? | **G** | ? | ? | ? | ? |

# Example: forward checking

**Map coloring**



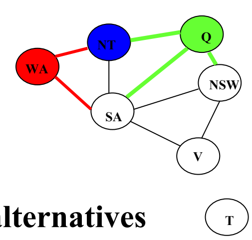

**Set: Q=Green**

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | G B | R G B | R G B | R G B | G B | R G B |
| *Q=Green* | **R** | B | **G** | R B | R G B | B | R G B |



# Example: forward checking

**Map coloring**

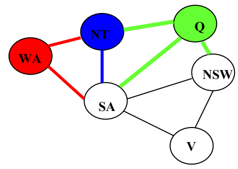



Infer: **Exhaustions of alternatives**

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | G B | R G B | R G B | R G B | G B | R G B |
| *Q=Green* | **R** | B | **G** | R B | R G B | B | R G B |
| *Infer NT* | **R** | **B** | **G** | ? | ? | ? | ? |

# Example: forward checking

**Map coloring**




Infer: **Exhaustions of alternatives**

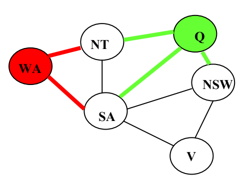
T

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | G B | R G B | R G B | R G B | G B | R G B |
| *Q=Green* | **R** | B | **G** | R B | R G B | B | R G B |
| *Infer NT* | **R** | **B** | **G** | R B | R G B | ! | R G B |



# Example: arc consistency

**Map coloring**



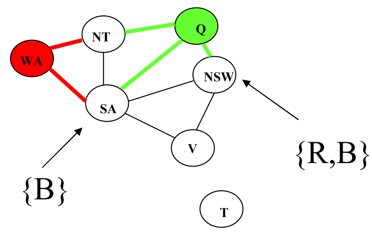

**Set: WA=Red**
**Set: Q=Green**

T

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | G B | R G B | R G B | R G B | G B | R G B |
| *Q=Green* | **R** | B | **G** | R B | R G B | B | R G B |

# Example: arc consistency

**Map coloring**



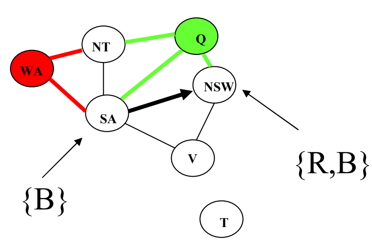

**Set: WA=Red**
**Set: Q=Green**

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | G B | R G B | R G B | R G B | G B | R G B |
| *Q=Green* | **R** | B | **G** | R B | R G B | B | R G B |



# Example: arc consistency

**Map coloring**




SA=B
NSW=R

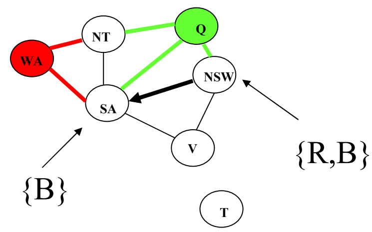
Consistent assignment

**Set: WA=Red**
**Set: Q=Green**

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | G B | R G B | R G B | R G B | G B | R G B |
| *Q=Green* | **R** | B | **G** | R B | R G B | B | R G B |

# Example: arc consistency

**Map coloring**




NSW=R
SA=B

**Consistent assignment**

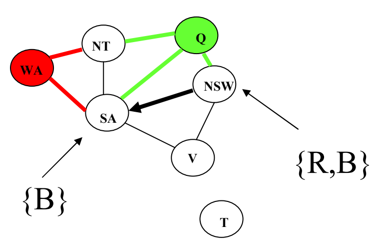
**Set: WA=Red**
**Set: Q=Green**

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | G B | R G B | R G B | R G B | G B | R G B |
| *Q=Green* | **R** | B | **G** | R B | R G B | B | R G B |



# Example: arc consistency

**Map coloring**




NSW=B
SA=!

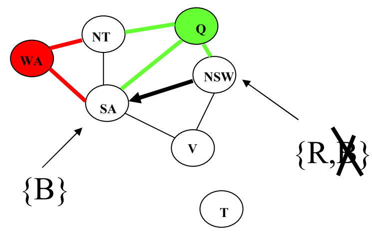
**Inconsistent assignment**

**Set: WA=Red**
**Set: Q=Green**

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | G B | R G B | R G B | R G B | G B | R G B |
| *Q=Green* | **R** | B | **G** | R B | R G B | B | R G B |

# Example: arc consistency

**Map coloring**




NSW=B
SA=!
**Inconsistent assignment**

**Set: WA=Red**
**Set: Q=Green**

| *vars* | WA | NT | Q | NSW | V | SA | T |
|---|---|---|---|---|---|---|---|
| *domain* | R G B | R G B | R G B | R G B | R G B | R G B | R G B |
| *WA=Red* | **R** | G B | R G B | R G B | R G B | G B | R G B |
| *Q=Green* | **R** | B | **G** | R B | R G B | B | R G B |
| | | | | **R** | | | |



# Heuristics for CSPs

**CSP** searches the space in the depth-first manner.
But we still can choose:

- **Which variable to assign next?**
- **Which value to choose first?**

**Heuristics**

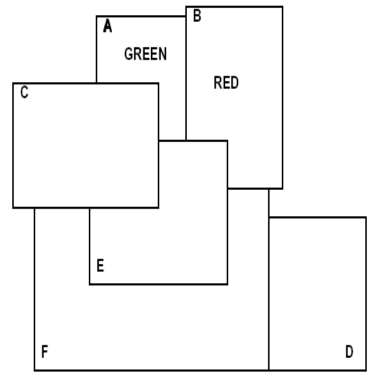
- **Most constrained variable**
  – Which variable is likely to become a bottleneck?
- **Least constraining value**
  – Which value gives us more flexibility later?

# Heuristics for CSP

Examples: **map coloring**

**Heuristics**

- **Most constrained variable**
  - ?
- **Least constraining value**
  - ?



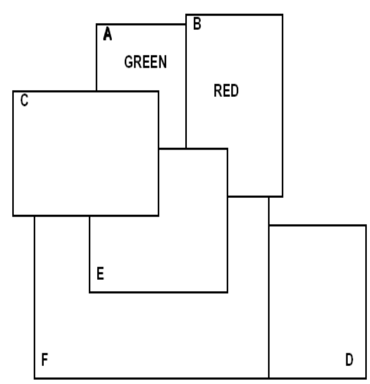



# Heuristics for CSP

Examples: **map coloring**

**Heuristics**

- **Most constrained variable**
  - Country E is the most constrained one (cannot use Red, Green)
- **Least constraining value**
  - ?

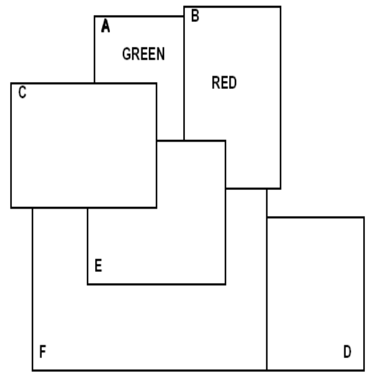


# Heuristics for CSP

Examples: **map coloring**

**Heuristics**

- **Most constrained variable**
  – Country E is the most constrained one (cannot use Red, Green)
- **Least constraining value**
  – Assume we have chosen variable C
  – What color is the least constraining color?



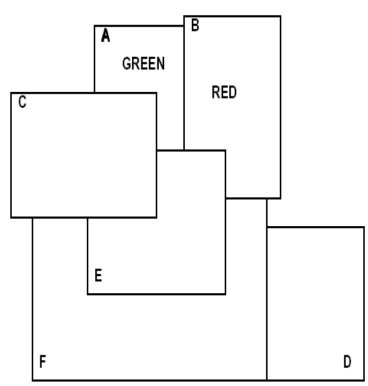



# Heuristics for CSP

Examples: **map coloring**

**Heuristics**

- **Most constrained variable**
  – Country E is the most constrained one (cannot use Red, Green)
- **Least constraining value**
  – Assume we have chosen variable C
  – Red is the least constraining valid color for the future




CS 1571 Intro to AI M. Hauskrecht

# Finding optimal configurations



## Search for the optimal configuration

**Constrain satisfaction problem:**

**Objective: find a configuration that satisfies all constraints**

**Optimal configuration problem:**

**Objective: find the best configuration**

**The quality of a configuration: is** defined by some quality measure that reflects our **preference towards each configuration** (or state)

**Our goal:** optimize the configuration according to the quality measure also referred to as objective function

## Search for the optimal configuration

**If the space of configurations we search among is**

- **Discrete or finite**
  - **then it is a combinatorial optimization problem**
- **Continuous**
  - **then it is a parametric optimization problem**

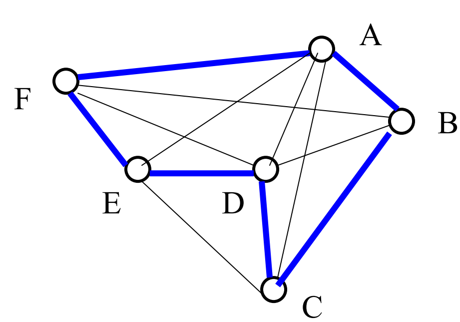
In the following we cover combinatorial optimization problems
Parametric optimization will covered next lecture.



## Example: Traveling salesman problem

**Problem:**

- A graph with distances

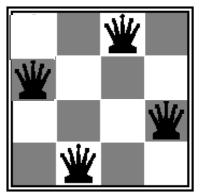



- **Goal:** find the shortest tour which visits every city once and returns to the start

  An example of a valid **tour:** ABCDEF

# Example: N queens

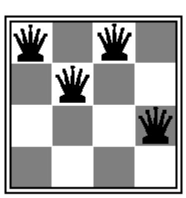
- A CSP problem
- Is it possible to formulate the problem as an optimal configuration search problem ?



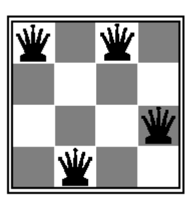


# Example: N queens

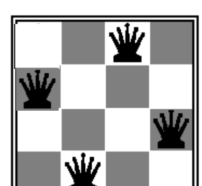
- A CSP problem
- Is it possible to formulate the problem as an optimal configuration search problem ? **Yes.**
- **The quality of a configuration in a CSP** can be measured by the number of violated constraints
- **Solving:** minimize the number of constraint violations



**# of violations =3**



**# of violations =1**



**# of violations =0**

## Iterative optimization methods

- Searching systematically for the best configuration with the **DFS** may not be the best solution
- Worst case running time:
  - Exponential in the number of variables
- Solutions to **large 'optimal' configuration** problems are often found using iterative optimization methods

- **Methods:**
  - **Hill climbing**
  - **Simulated Annealing**
  - **Genetic algorithms**



## Iterative optimization methods

**Properties:**

- **Search the space of "complete" configurations**
- **Take advantage of local moves**
  - Operators make "local" changes to "complete" configurations
- **Keep track of just one state (the current state)**
  - no memory of past states
  - **!!! No search tree is necessary !!!**

# Example: N-queens

- **"Local" operators for generating the next state:**
  - Select a variable (a queen)
  - Reallocate its position



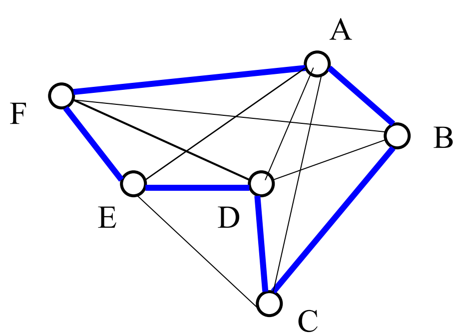
# Example: Traveling salesman problem

**"Local" operator for generating the next state:**

- divide the existing tour into two parts,
- reconnect the two parts in the opposite order

**Example:**

ABCDEF



# Example: Traveling salesman problem

**"Local" operator for generating the next state:**

- divide the existing tour into two parts,
- reconnect the two parts in the opposite order

**Example:**

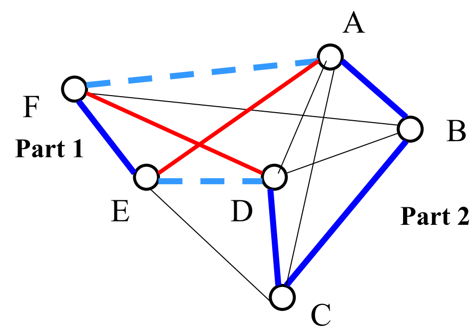
ABCDEF

↓

ABCD | EF |

↓

ABCDFE

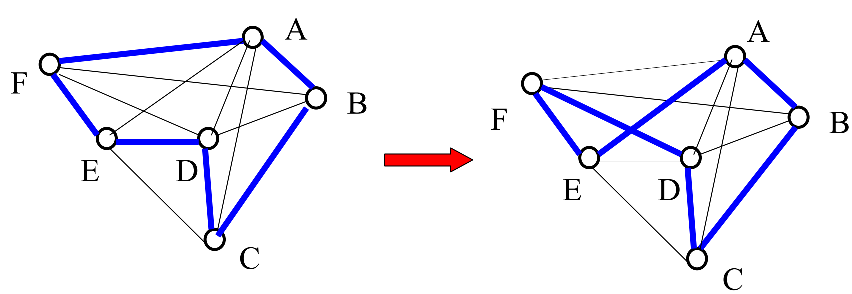





# Example: Traveling salesman problem

**"Local" operator:**

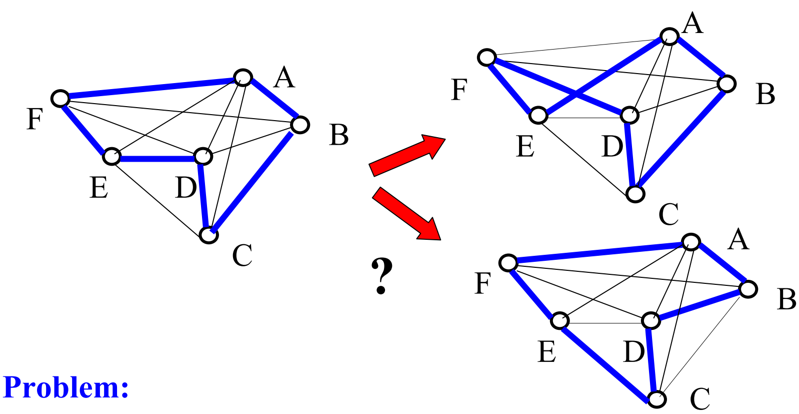
– generates the next configuration (state)



# Searching the configuration space

## Search algorithms

- keep only one configuration (the current configuration)




## Problem:

- How to decide about which operator to apply?



# Search algorithms

**Two strategies to choose the configuration (state) to be visited next:**

- **Hill climbing**
- **Simulated annealing**

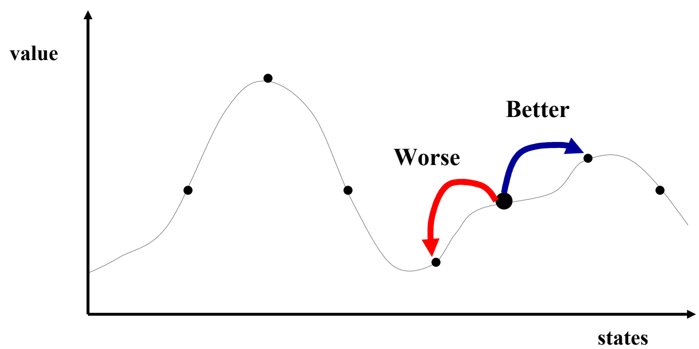
- Later: Extensions to multiple current states:
  - **Genetic algorithms**

- **Note:** Maximization is inverse of the minimization

$$\min f(X) \Leftrightarrow \max\left[-f(X)\right]$$

# Hill climbing

- Look around at states in the local neighborhood and choose the one with the best value
- Assume: we want to maximize the






# Hill climbing

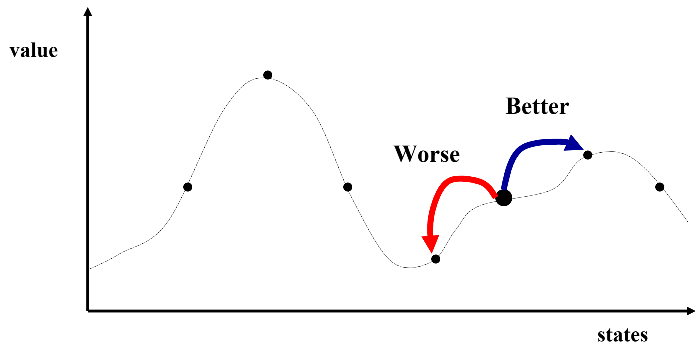
- Always choose the next best successor state
- Stop when no improvement possible

```
function HILL-CLIMBING(problem) returns a solution state
  inputs: problem, a problem
  static: current, a node
          next, a node

  current ← MAKE-NODE(INITIAL-STATE[problem])
  loop do
     next ← a highest-valued successor of current
     if VALUE[next] < VALUE[current] then return current
     current ← next
  end
```

# Hill climbing

- Look around at states in the local neighborhood and choose the one with the best value

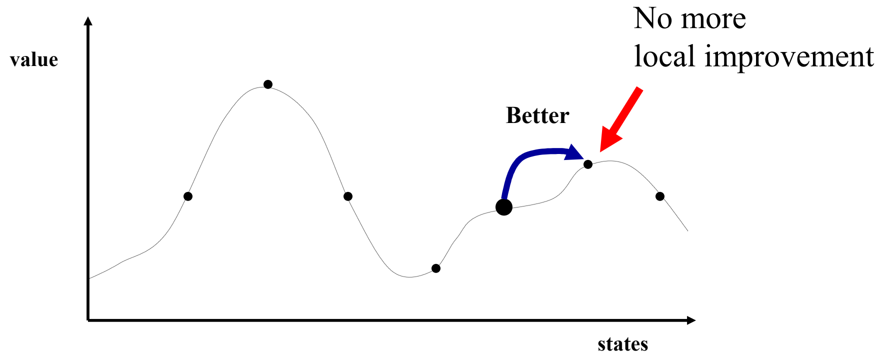



- **What can go wrong?**



# Hill climbing

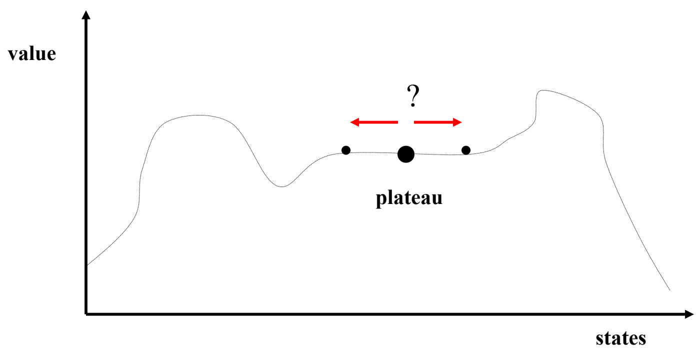
- Hill climbing can get trapped in the local optimum




- **What can go wrong?**

# Hill climbing

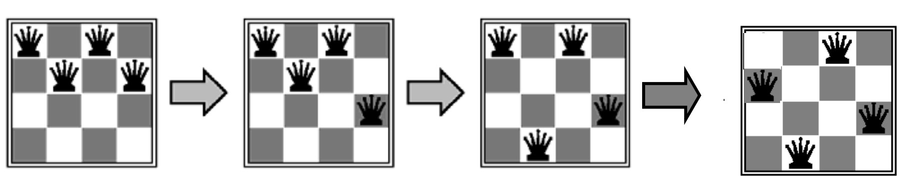
- Hill climbing can get clueless on plateaus






# Hill climbing and n-queens

- The quality of a configuration is given by the number of constraints violated
- **Then: Hill climbing** reduces the number of constraints
- **Min-conflict strategy (heuristic):**
  - Choose randomly a variable with conflicts
  - Choose its value such that it violates the fewest constraints



Success !! But not always!!! The local optima problem!!!