CS 1571 Introduction to AI
Lecture 22

Inferences in Bayesian belief networks

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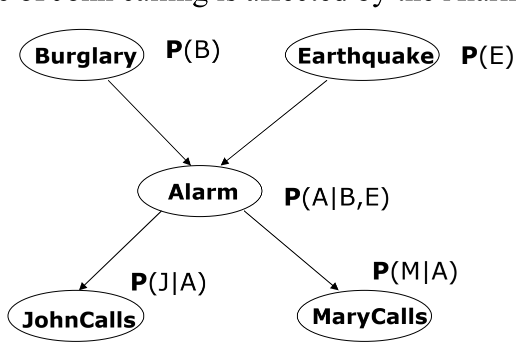
Bayesian belief network.

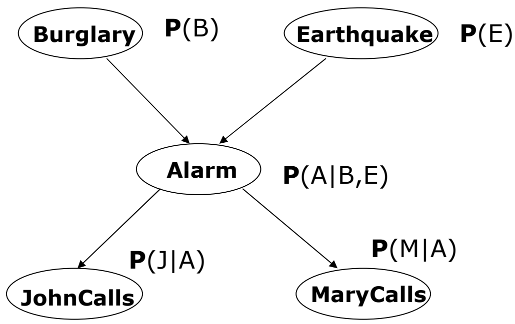
1. **Directed acyclic graph**
   - **Nodes** = random variables
     Burglary, Earthquake, Alarm, Mary calls and John calls
   - **Links** = direct (causal) dependencies between variables.
     The chance of Alarm is influenced by Earthquake, The chance of John calling is affected by the Alarm



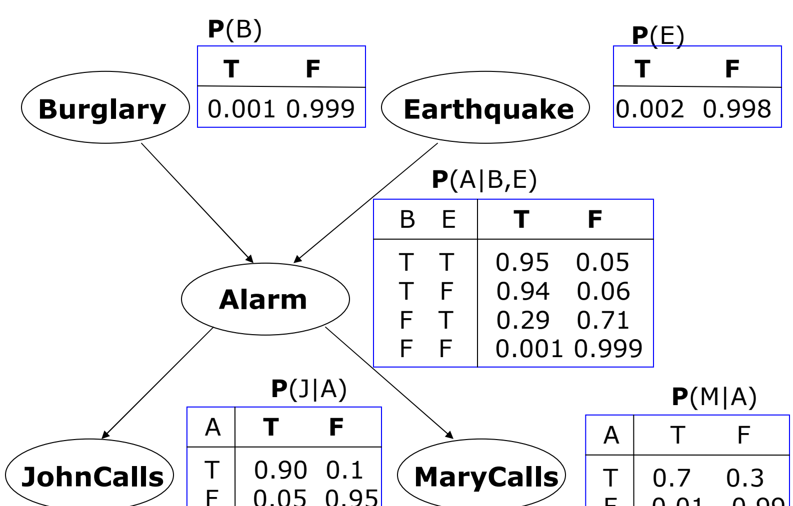
Bayesian belief network.

2. Local conditional distributions

- relate variables and their parents



Bayesian belief network.



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

P(J|A)

A	T	F
T	0.90	0.1
F	0.05	0.95

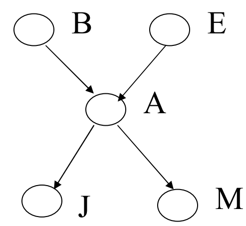
P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

Bayesian belief networks (general)

Two components: $B = (S, \Theta_S)$

- **Directed acyclic graph**
  - Nodes correspond to random variables
  - (Missing) links encode independences



- **Parameters**
  - Local conditional probability distributions for every variable-parent configuration

$$\mathbf{P}(X_i \mid pa(X_i))$$

Where:

$pa(X_i)$ - stand for parents of X_i

P(A|B,E)

B	E	**T**	**F**
T	T	0.95	0.05
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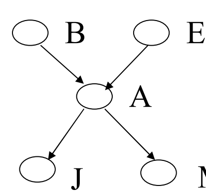
Full joint distribution in BBNs

Full joint distribution is defined in terms of local conditional distributions (obtained via the chain rule):

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

Example:



Assume the following assignment of values to random variables

$B = T, E = T, A = T, J = T, M = F$

Then its probability is:

$$P(B = T, E = T, A = T, J = T, M = F) =$$

$$P(B = T) P(E = T) P(A = T \mid B = T, E = T) P(J = T \mid A = T) P(M = F \mid A = T)$$

Parameter complexity problem

- In the BBN the **full joint distribution** is defined as:

$$\mathbf{P}(X_1, X_2, \ldots, X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

- **What did we save?**

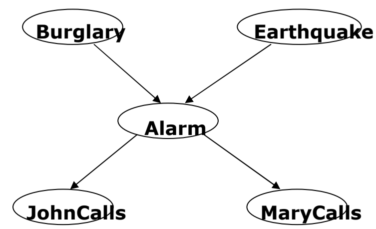
Alarm example: 5 binary (True, False) variables

# of parameters of the full joint:

$$2^5 = 32$$

One parameter is for free:

$$2^5 - 1 = 31$$



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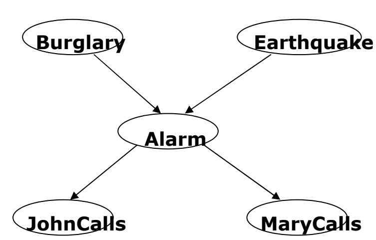
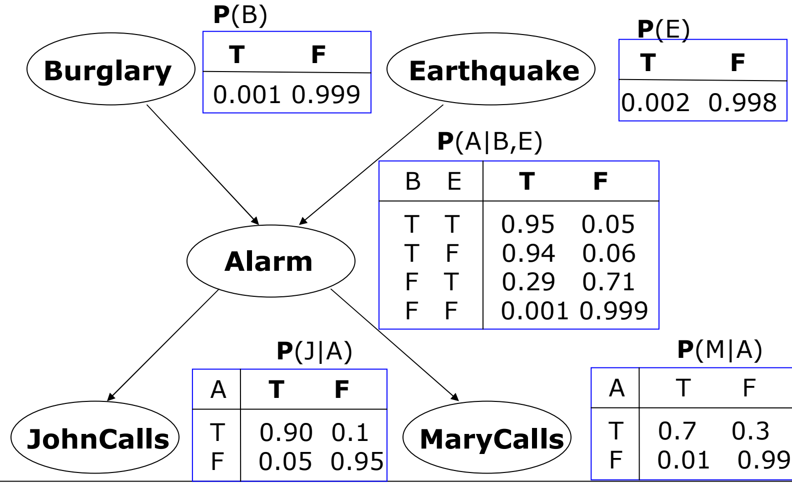


Bayesian belief network.

- In the BBN the **full joint distribution** is expressed using a set of local conditional distributions



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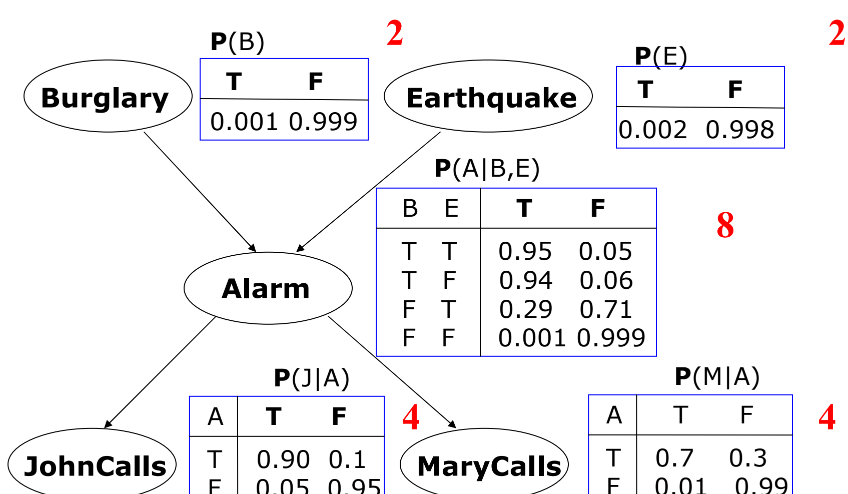

P(B) — 2

T	F
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P(E) — 2

T	F
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P(A|B,E) — 8

B	E	T	F
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P(J|A) — 4

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# of parameters of the BBN:

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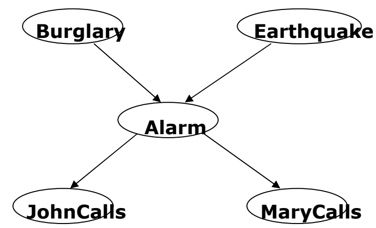

One parameter in every conditional is for free:

?

Parameter complexity problem

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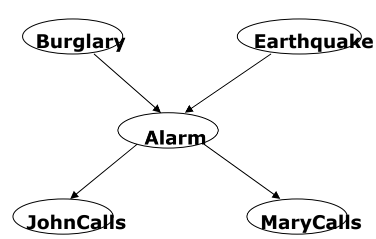
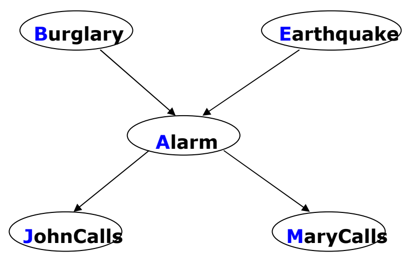


One parameter in every conditional is for free:

$$2^2 + 2(2) + 2(1) = 10$$

Inference in Bayesian network

- **Bad news:**
  - Exact inference problem in BBNs is NP-hard (Cooper)
  - Approximate inference is NP-hard (Dagum, Luby)
- **But** very often we can achieve significant improvements
- Assume our Alarm network



- Assume we want to compute: $P(J = T)$

Inference in Bayesian networks

Computing: $P(J = T)$

Approach 1. Blind approach.

- Sum out all un-instantiated variables from the full joint,
- express the joint distribution as a product of conditionals

$$P(J = T) =$$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(B = b, E = e, A = a, J = T, M = m)$$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(J = T \mid A = a) P(M = m \mid A = a) P(A = a \mid B = b, E = e) P(B = b) P(E = e)$$

Computational cost:

Number of additions: ?

Number of products: ?

Inference in Bayesian networks

Computing: $P(J=T)$

Approach 1. Blind approach.

- Sum out all un-instantiated variables from the full joint,
- express the joint distribution as a product of conditionals

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$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F} P(B=b,E=e,A=a,J=T,M=m)$$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F} P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

Computational cost:

Number of additions: 15

Number of products: 16*4=64

Inference in Bayesian networks

Approach 2. Interleave sums and products

- Combines sums and product in a smart way (multiplications by constants can be taken out of the sum)

$$P(J=T) =$$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F} P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

$$= \sum_{b\in T,F}\sum_{a\in T.F}\sum_{m\in T,F} P(J=T\mid A=a)P(M=m\mid A=a)P(B=b)[\sum_{e\in T,F} P(A=a\mid B=b,E=e)P(E=e)]$$

$$= \sum_{a\in T,F} P(J=T\mid A=a)[\sum_{m\in T,F} P(M=m\mid A=a)][\sum_{b\in T,F} P(B=b)[\sum_{e\in T,F} P(A=a\mid B=b,E=e)P(E=e)]]$$

Computational cost:

Number of additions: 1+2*[1+1+2*1]=**?**

Number of products: 2*[2+2*(1+2*1)]=?

Inference in Bayesian networks

Approach 2. Interleave sums and products

- Combines sums and product in a smart way (multiplications by constants can be taken out of the sum)

$$P(J=T)=$$

$$=\sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F}P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

$$=\sum_{b\in T,F}\sum_{a\in T.F}\sum_{m\in T,F}P(J=T\mid A=a)P(M=m\mid A=a)P(B=b)[\sum_{e\in T,F}P(A=a\mid B=b,E=e)P(E=e)]$$

$$=\sum_{a\in T,F}P(J=T\mid A=a)[\sum_{m\in T,F}P(M=m\mid A=a)][\sum_{b\in T,F}P(B=b)[\sum_{e\in T,F}P(A=a\mid B=b,E=e)P(E=e)]]$$

Computational cost:

Number of additions: 1+2*[1+1+2*1]=**9**

Number of products: 2*[2+2*(1+2*1)]=16

Variable elimination

- **Variable elimination:**
  - Similar idea but interleave sum and products one variable at the time during inference
  - E.g. Query $P(J=T)$ requires to eliminate A,B,E,M and this can be done in different order

$$P(J=T)=$$

$$=\sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F}P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

Variable elimination

Assume order: M, E, B,A to calculate $P(J=T)$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F} P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F} P(J=T\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)\left[\sum_{m\in T,F} P(M=m\mid A=a)\right]$$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F} P(J=T\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e) \quad 1$$

$$= \sum_{a\in T,F}\sum_{b\in T,F} P(J=T\mid A=a)P(B=b)\left[\sum_{e\in T,F} P(A=a\mid B=b,E=e)P(E=e)\right]$$

$$= \sum_{a\in T,F}\sum_{b\in T,F} P(J=T\mid A=a)P(B=b)\,\tau_1(A=a,B=b)$$

$$= \sum_{a\in T,F} P(J=T\mid A=a)\left[\sum_{b\in T,F} P(B=b)\,\tau_1(A=a,B=b)\right]$$

$$= \sum_{a\in T,F} P(J=T\mid A=a)\quad \tau_2(A=a)$$

Inference in Bayesian network

- **Exact inference algorithms:**
  - **Variable elimination** (Book)
  - **Recursive decomposition** (Cooper, Darwiche)
  - Symbolic inference (D'Ambrosio)
  - Belief propagation algorithm (Pearl)
  - **Clustering and joint tree approach (Lauritzen, Spiegelhalter)** (Book)
  - Arc reversal (Olmsted, Schachter)
- **Approximate inference algorithms:**
  - **Monte Carlo methods:** (Book)
    - Forward sampling, Likelihood sampling
  - Variational methods

Monte Carlo approaches

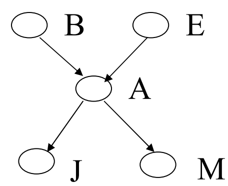
- **MC approximation**:
  - The probability is approximated using sample frequencies
  - **Example:**

$$\tilde{P}(B=T, J=T) = \frac{N_{B=T,J=T}}{N}$$

$N_{B=T,J=T}$ ← *# samples with B = T, J = T*

N ← *total # samples*

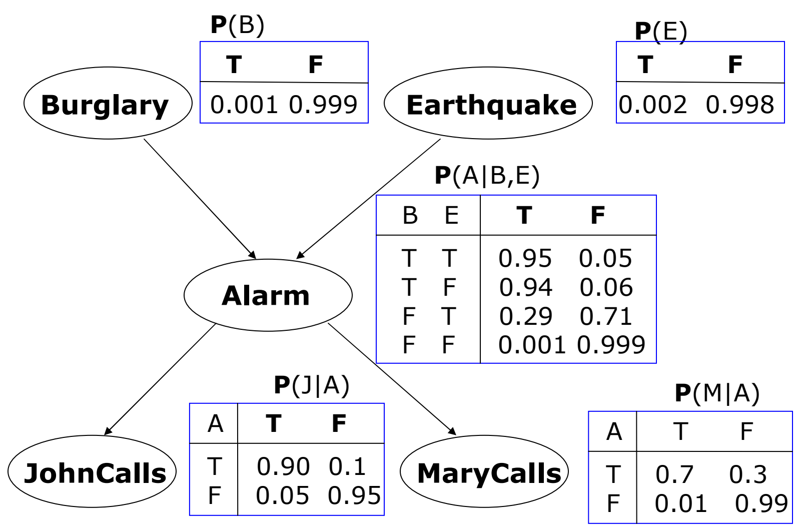
- **BBN sampling:**



Generate sample in a top down manner, following the links

- **One sample gives one assignment of values to all variables**

BBN sampling example



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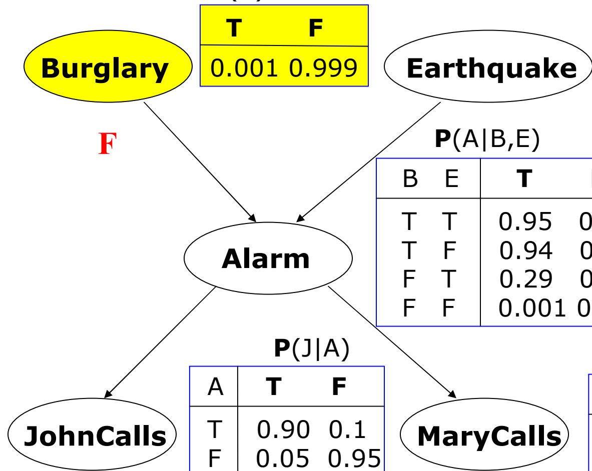

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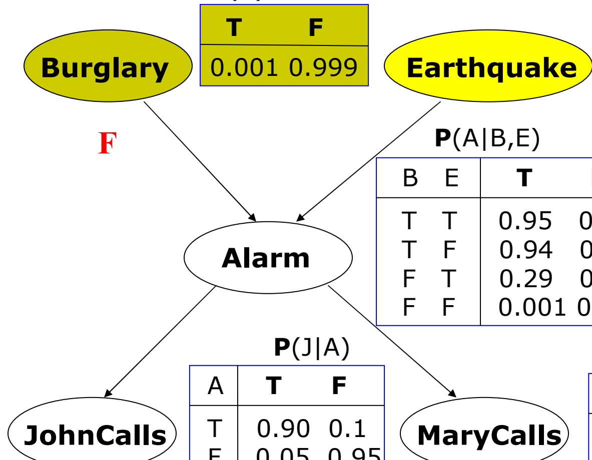

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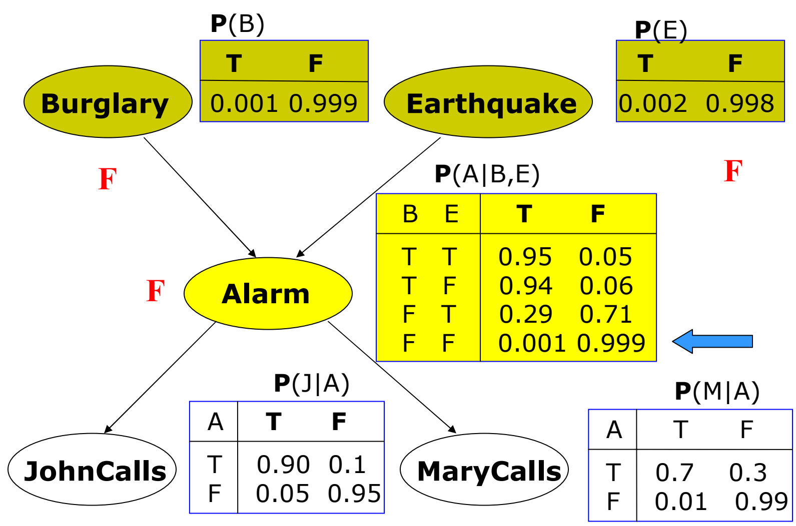

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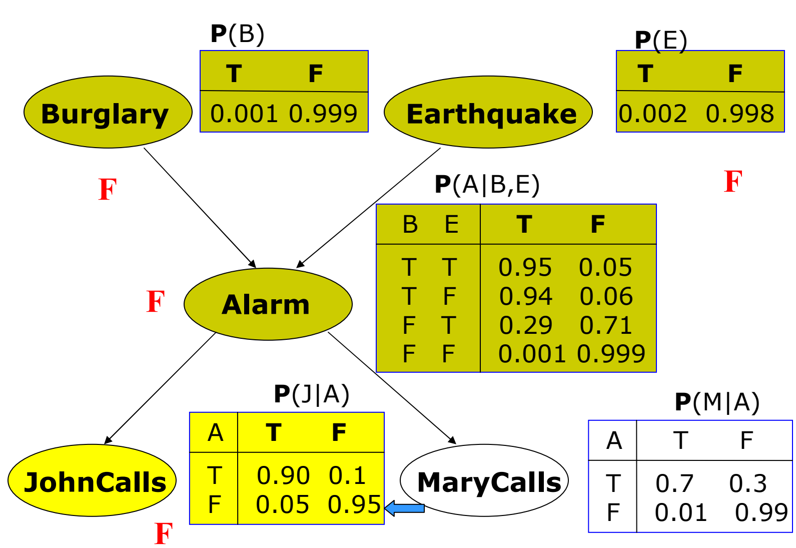

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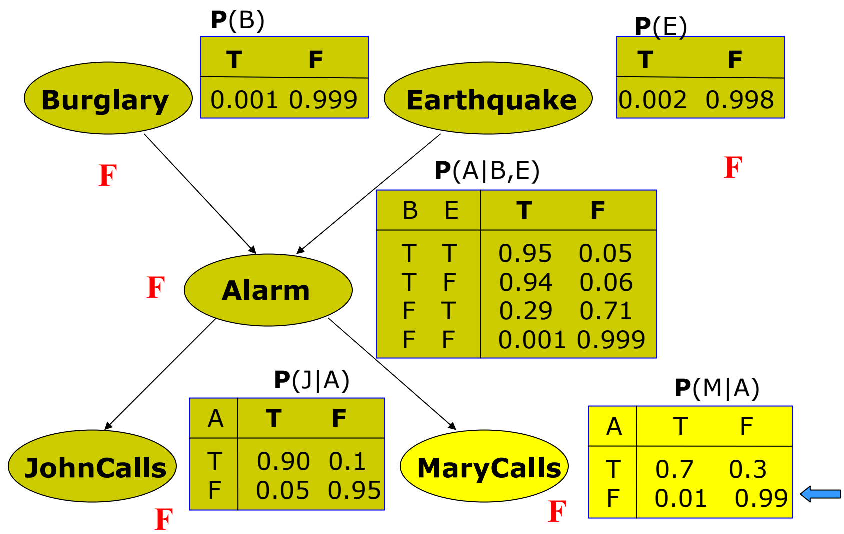

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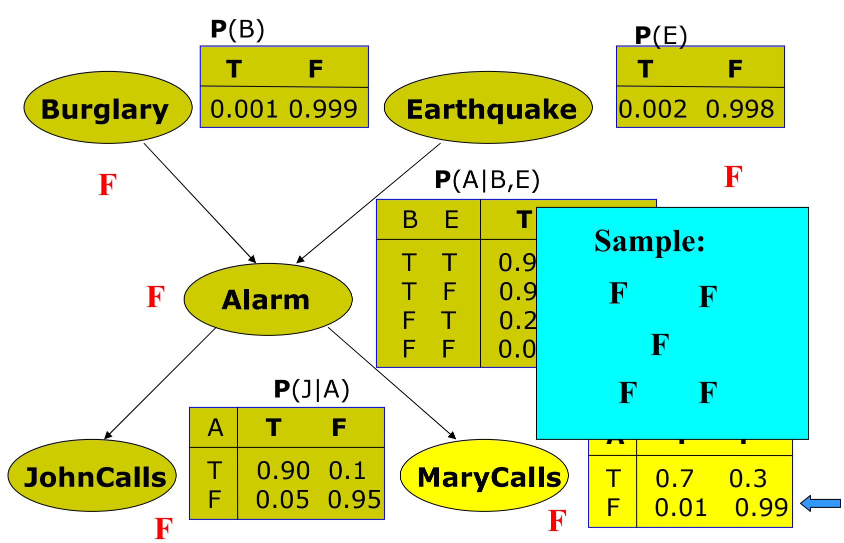


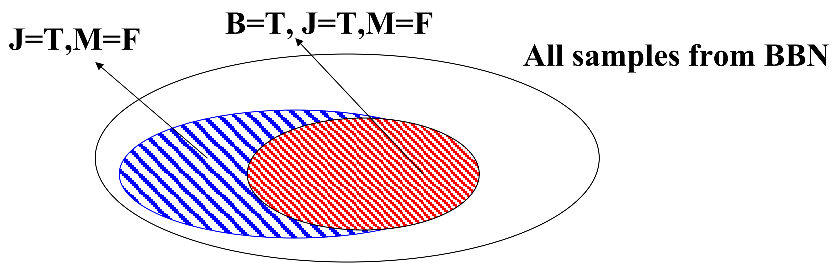
Monte Carlo approaches

- **MC approximation of conditional probabilities**:
  - The probability is approximated using sample frequencies
  - **Example:**

$\# samples\ \ with\ \ B=T, J=T, M=F$

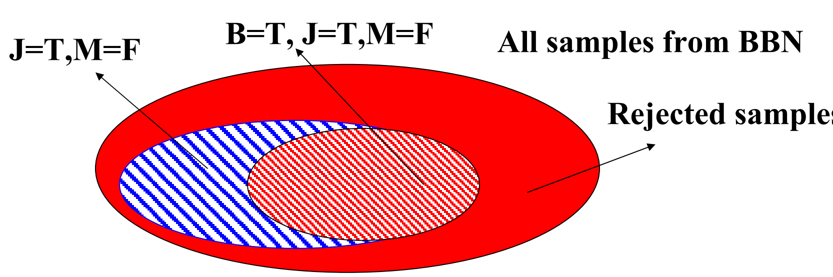
$$\tilde{P}(B=T \mid J=T, M=F) = \frac{N_{B=T,J=T,M=F}}{N_{J=T,M=F}}$$

$\# samples\ \ with\ \ J=T, M=F$



Monte Carlo approaches

- **Rejection sampling**
  - Generate samples from the full joint by sampling BBN
  - Use only samples that agree with the condition, the remaining samples are rejected
- **Problem:** many samples can be rejected



Likelihood weighting

Idea: generate only samples consistent with an evidence (or conditioning event)

- Benefit: Avoids inefficiencies of rejection sampling

Problem:

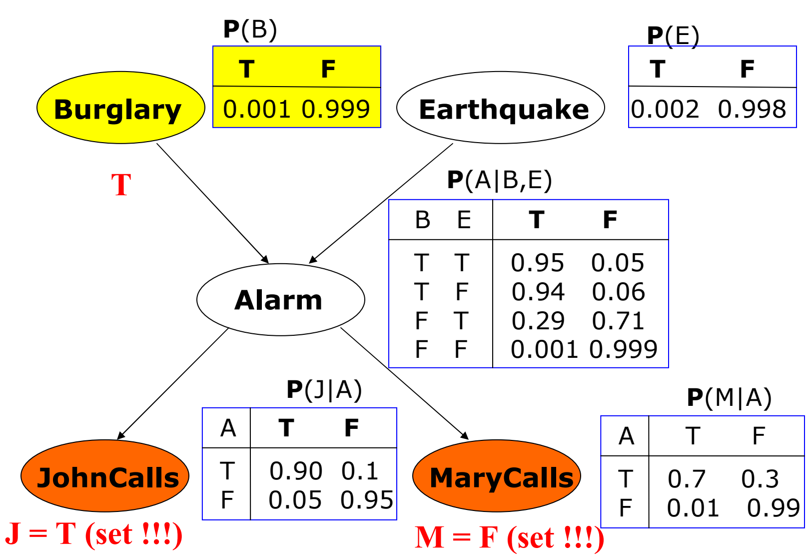
- **the distribution generated by enforcing the conditioning variables to set values is biased**
- simple counts are not sufficient to estimate the probabilities

Solution:

- **With every sample keep a weight with which it should count towards the estimate**

$$\tilde{P}(B=T \mid J=T, M=F) = \frac{\sum_{\text{samples with } B=T, M=F \text{ and } J=T} w_{B=T|J=T,M=F}}{\sum_{\text{samples with any value of } B \text{ and } J=T, M=F} w_{B=x|J=T,M=F}}$$

BBN likelihood weighting example



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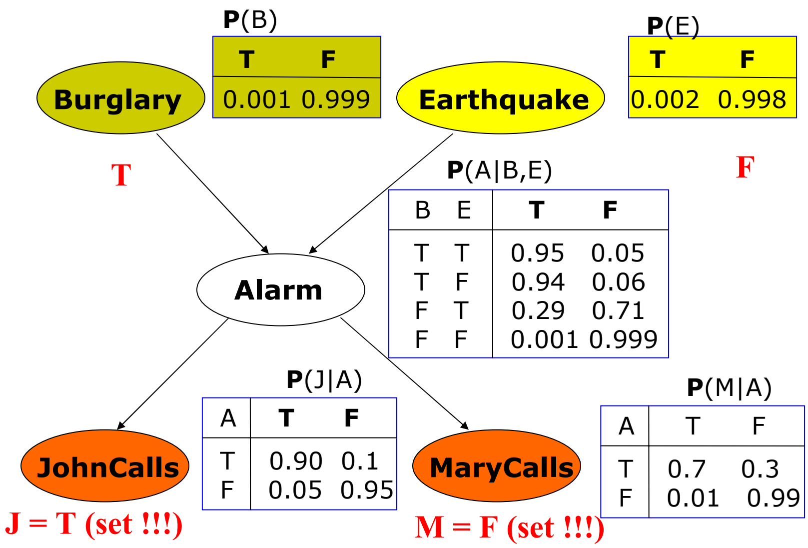

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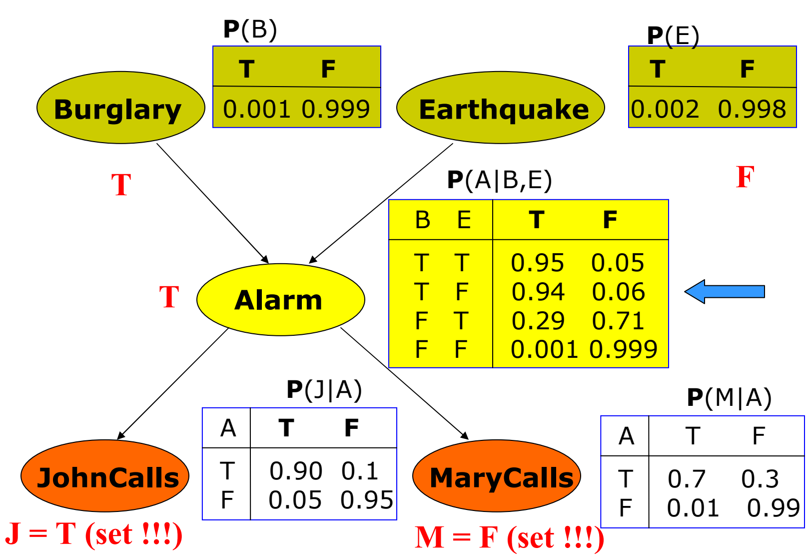
BBN likelihood weighting example



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

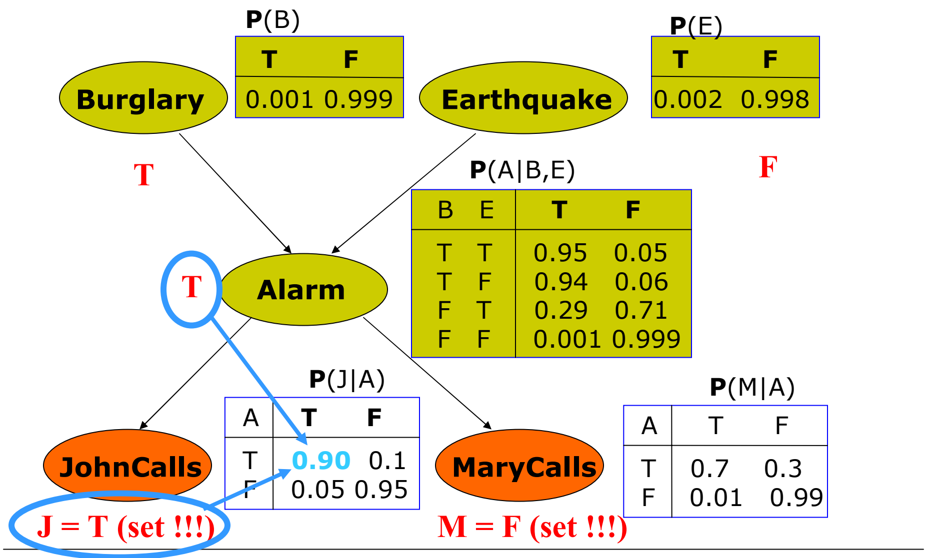
P(J|A)

A	T	F
T	0.90	0.1
F	0.05	0.95

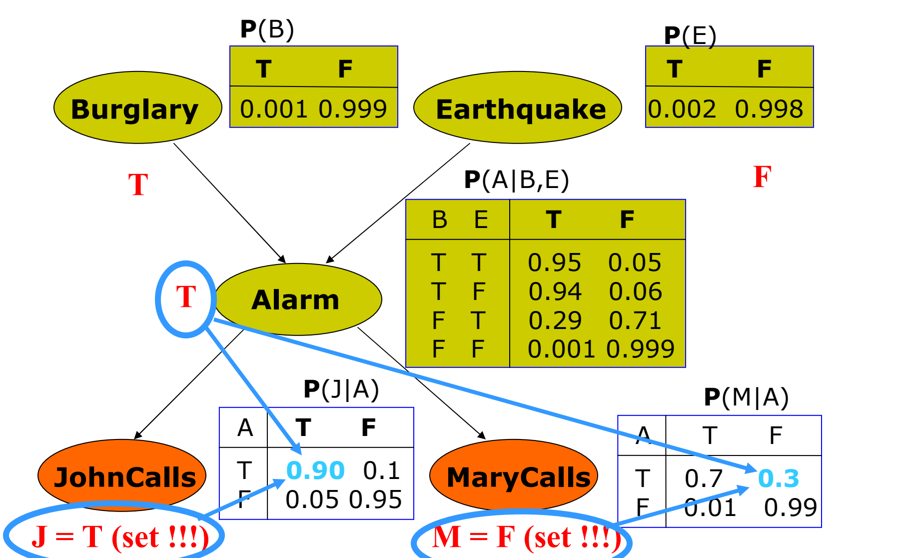
P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

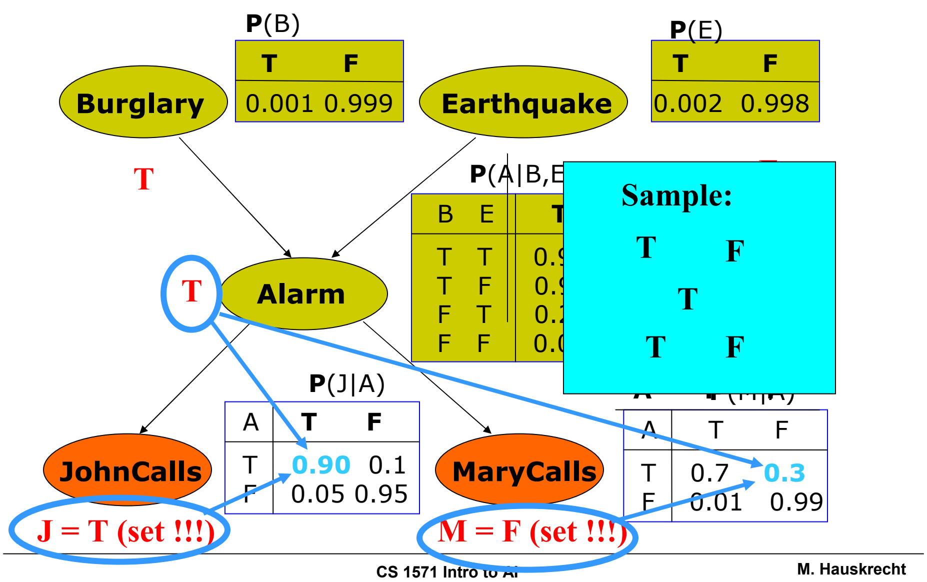
BBN likelihood weighting example



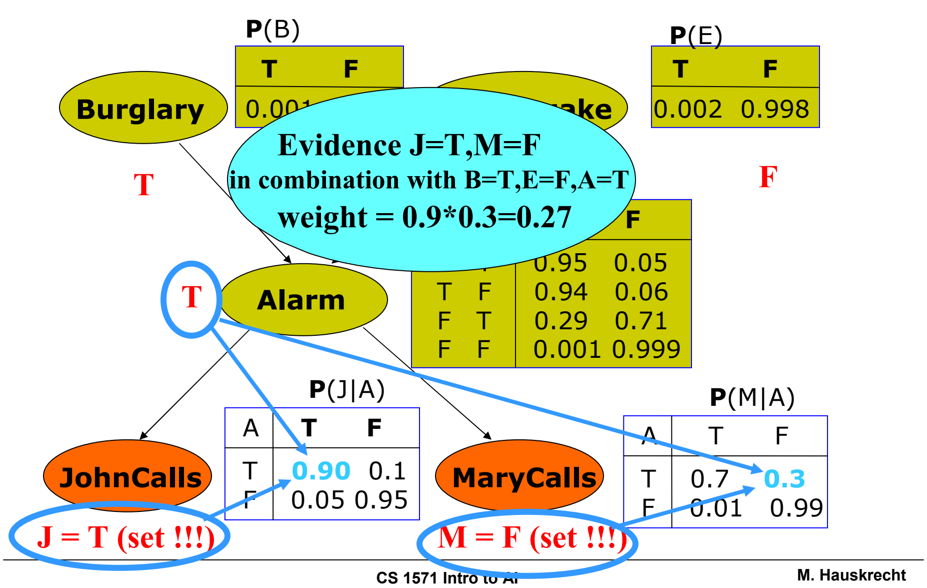
BBN likelihood weighting example



BBN likelihood weighting example

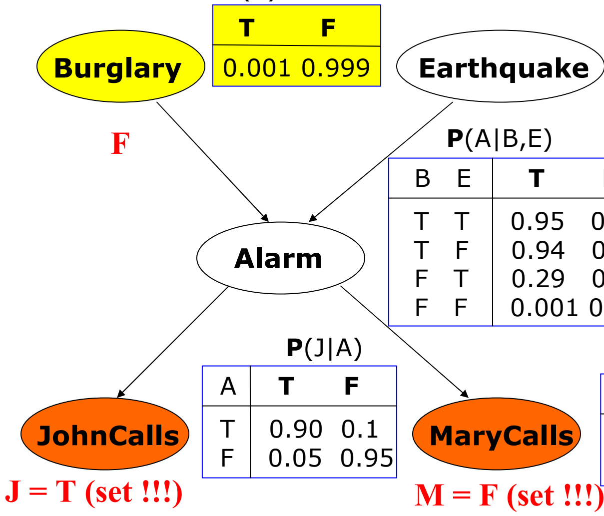


BBN likelihood weighting example



BBN likelihood weighting example

Second sample



P(B)

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0.002	0.998

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T	T	0.95	0.05
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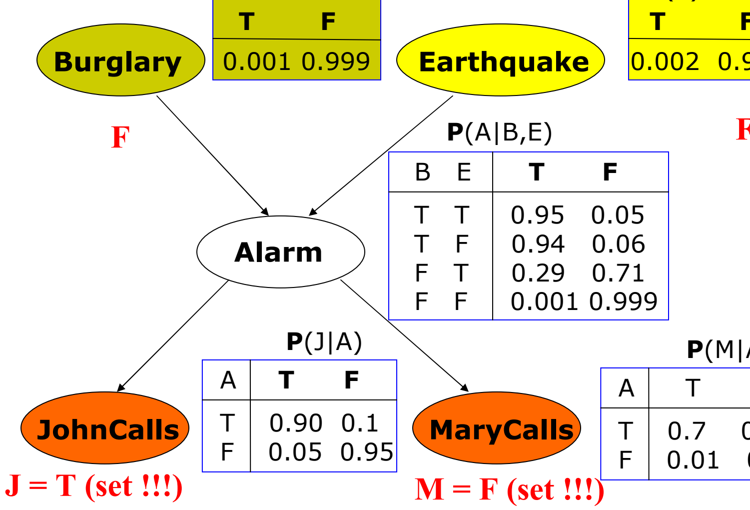
A	T	F
T	0.7	0.3
F	0.01	0.99

BBN likelihood weighting example

Second sample



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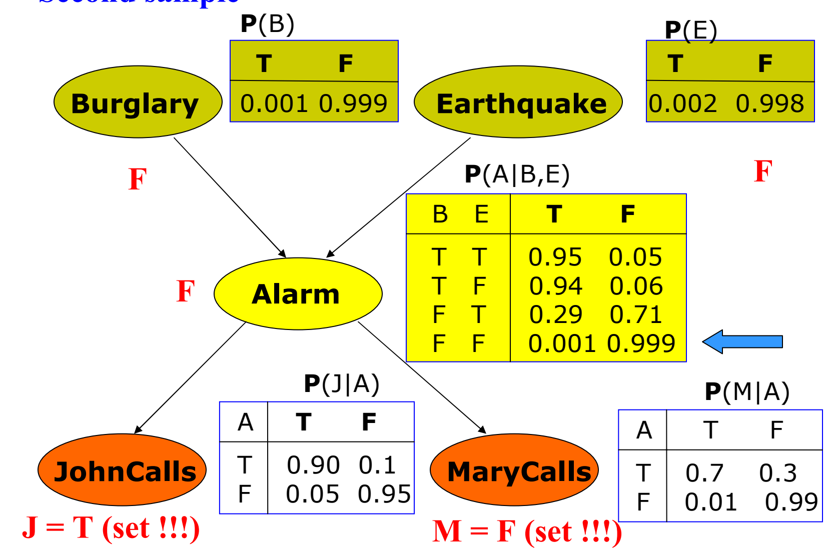
BBN likelihood weighting example

Second sample



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T	T	0.95	0.05
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F	T	0.29	0.71
F	F	0.001	0.999

P(J|A)

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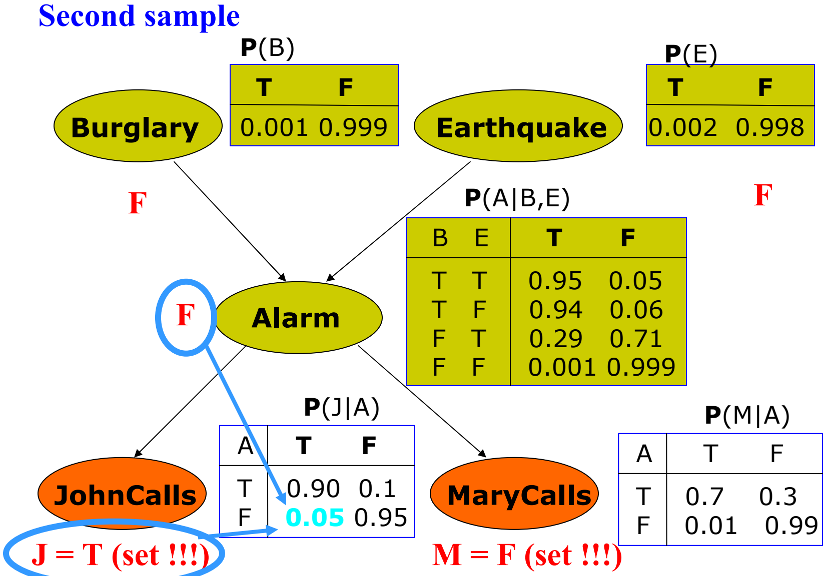
P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

BBN likelihood weighting example

Second sample



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

P(J|A)

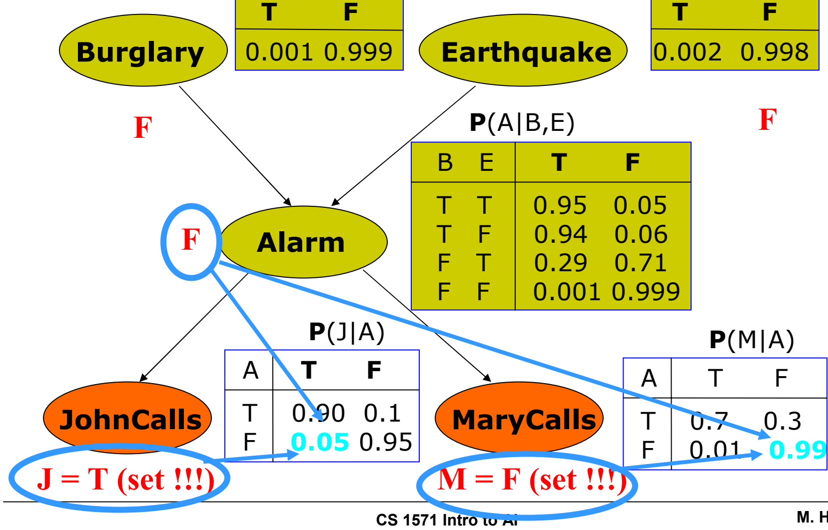
A	T	F
T	0.90	0.1
F	0.05	0.95

P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

BBN likelihood weighting example

Second sample



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

P(J|A)

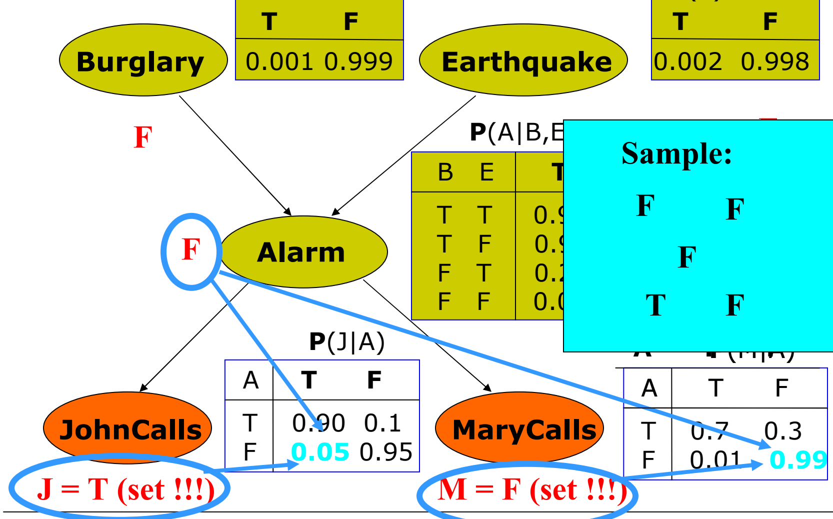
A	T	F
T	0.90	0.1
F	0.05	0.95

P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

BBN likelihood weighting example

Second sample



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E

B	E	T
T	T	0.9
T	F	0.9
F	T	0.2
F	F	0.0

P(J|A)

A	T	F
T	0.90	0.1
F	0.05	0.95

A	T	F
T	0.7	0.3
F	0.01	0.99

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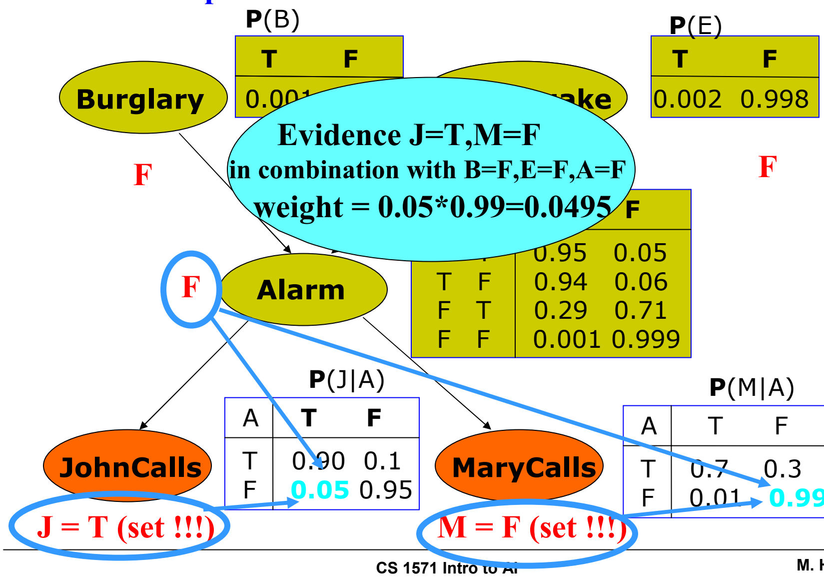
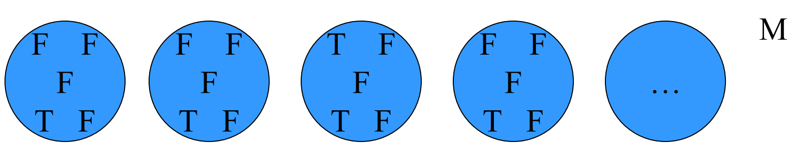
BBN likelihood weighting example

Second sample



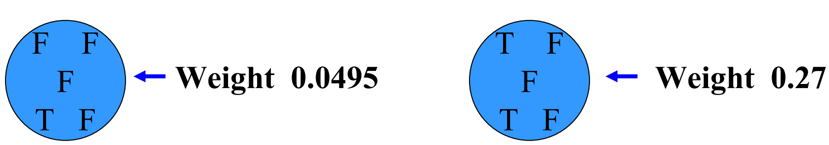
Likelihood weighting

- Assume we have generated the following M samples:



How to make the samples consistent?

Weight each sample by probability with which it agrees with the conditioning evidence P(e).



Monte Carlo approaches

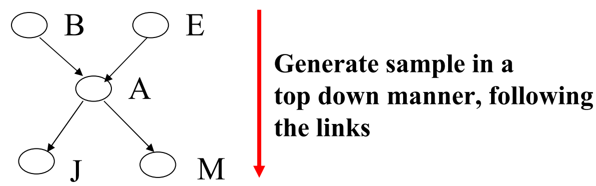
- **MC approximation**:
  - The probability is approximated using sample frequencies
  - **Example:**

$$\tilde{P}(B=T, J=T) = \frac{N_{B=T,J=T}}{N}$$

$N_{B=T,J=T}$ ← $\#\, samples\;\; with\;\; B=T, J=T$

N ← $total\;\; \#\, samples$

- **BBN sampling:**



- **One sample gives one assignment of values to all variables**