CS 1571 Introduction to AI
Lecture 14

First-order logic

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Administration announcements

Midterm:

- **Thursday, October 28, 2010**
- **In-class**
- **Closed book**

Limitations of propositional logic

World we want to represent and reason about consists of a number of objects with variety of properties and relations among them

Propositional logic:

- Represents statements about the world without reflecting this structure and without modeling these entities explicitly

Consequence:

- some knowledge is hard or impossible to encode in the propositional logic.
- Two cases that are hard to represent:
  - **Statements about similar objects, relations**
  - **Statements referring to groups of objects**.

Limitations of propositional logic

- **Statements about similar objects and relations needs to be enumerated**
- **Example:** Seniority of people domain

  For inferences we need:

  John is older than Mary $\wedge$ *Mary is older than Paul*
  $\Rightarrow$ *John is older than Paul*

  Jane is older than Mary $\wedge$ *Mary is older than Paul*
  $\Rightarrow$ *Jane is older than Paul*

- **Problem:** if we have many people and facts about their seniority we need represent many rules like this to allow inferences
- **Possible solution: ??**

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- **Problem:** if we have many people and facts about their seniority we need represent many rules like this to allow inferences
- **Possible solution: introduce variables**

  PersA *is older than* ***PersB*** $\wedge$ ***PersB*** *is older than* ***PersC***
  $\Rightarrow$ ***PersA*** *is older than* ***PersC***

Limitations of propositional logic

- **Statements referring to groups of objects require exhaustive enumeration of objects**
- **Example:**

  Assume we want to express *Every student likes vacation*

  Doing this in propositional logic would require to include statements about every student

  John likes vacation $\wedge$
  Mary likes vacation $\wedge$
  Ann likes vacation $\wedge$
  …

- **Solution:** Allow quantification in statements

First-order logic (FOL)

- More expressive than **propositional logic**

- **Eliminates deficiencies of PL by:**
  – Representing objects, their properties, relations and statements about them;
  – Introducing variables that refer to an arbitrary objects and can be substituted by a specific object
  – Introducing quantifiers allowing us to make statements over groups objects without the need to represent each of them separately

Logic

Logic is defined by**:**

- **A set of sentences**
  – A sentence is constructed from a set of primitives according to syntax rules.
- **A set of interpretations**
  – An interpretation gives a semantic to primitives. It associates primitives with objects, values in the real world.
- **The valuation (meaning) function** V
  – Assigns a truth value to a given sentence under some interpretation

$$V : \text{sentence} \times \text{interpretation} \rightarrow \{True, False\}$$

First-order logic. Syntax.

Term – a syntactic entity for representing objects

Terms in FOL:

- **Constant symbols:** represent specific objects
  - E.g. *John, France, car89*
- **Variables:** represent objects of a certain type (type = domain of discourse)
  - E.g. *x,y,z*
- **Functions** applied to one or more terms
  - E.g. *father-of (John)*
    father-of(father-of(John))

First order logic. Syntax.

Sentences in FOL:

- **Atomic sentences:**
  - **A predicate symbol** applied to 0 or more terms
    Examples:
    Red(car12),
    Sister(Amy, Jane);
    Manager(father-of(John));
  - t1 = t2 **equivalence** of terms
    Example:
    John = father-of(Peter)

First order logic. Syntax.

Sentences in FOL:

- **Complex sentences:**
- Assume ϕ, ψ are sentences in FOL. Then:
  - $(\phi \wedge \psi)$ $(\phi \vee \psi)$ $(\phi \Rightarrow \psi)$ $(\phi \Leftrightarrow \psi)$ $\neg \psi$ and
  - $\forall x\ \phi$ $\exists y\ \phi$

  are sentences

Symbols $\exists, \forall$

- stand for the **existential** and the **universal** quantifier

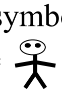
Semantics. Interpretation.

An interpretation ***I*** is defined by a **mapping** constants, predicates and function to the **domain of discourse D or relations on D**

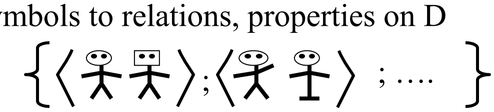
- **domain of discourse:** a set of objects in the world we represent and refer to;

An interpretation *I* maps:

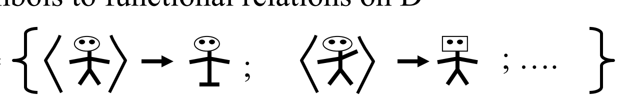
- Constant symbols to objects in D

  I(*John*) = 

- Predicate symbols to relations, properties on D

  I(*brother*) = 

- Function symbols to functional relations on D

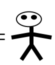
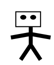
  I(*father-of*) = 

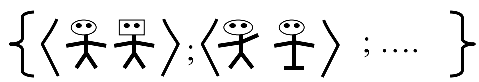
Semantics of sentences.

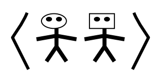
Meaning (evaluation) function:

V : sentence $\times$ interpretation $\rightarrow \{True\ , False\ \}$

A **predicate** *predicate(term-1, term-2, term-3, term-n)* is true for the interpretation I , iff the objects referred to by *term-1, term-2, term-3, term-n* are in the relation referred to by *predicate*

I(*John*) =  I(*Paul*) = 

I(*brother*) = 

brother(John, Paul) =  in I(*brother*)

V(*brother(John, Paul), I*) = ***True***

Semantics of sentences.

- **Equality** V(*term-1= term-2, I*) = ***True***
  Iff I(*term-1*) =I(*term-2*)
- **Boolean expressions: standard**

  E.g. V(*sentence-1* $\vee$ *sentence-2, I*) = ***True***
  Iff V(*sentence-1,I*)= ***True*** or V(*sentence-2,I*)= ***True***

- **Quantifications**

  V($\forall x\ \phi$, *I*) = ***True***
  substitution of *x* with *d*
  Iff for all $d \in D$ V(ϕ ,*I[x/d]*)= ***True***

  V($\exists x\ \phi$, *I*) = ***True***
  Iff there is a $d \in D$, s.t. V(ϕ ,*I[x/d]*)= ***True***

Sentences with quantifiers

- **Universal quantification**

  All Upitt students are smart

- **Assume the universe of discourse of x are Upitt students**

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- **Assume the universe of discourse of x are people**

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- **Assume the universe of discourse of x are people**

$$\forall x \; student(x) \wedge at(x, Upitt) \Rightarrow smart(x)$$

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Typically the universal quantifier connects with an implication

Sentences with quantifiers

- **Existential quantification**

Someone at CMU is smart

- **Assume the universe of discourse of x are CMU affiliates**

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Sentences with quantifiers

- **Existential quantification**

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- **Assume the universe of discourse of x are CMU affiliates**

$$\exists x \; smart(x)$$

- **Assume the universe of discourse of x are people**

$$\exists x \; at(x,CMU) \quad \wedge \; smart(x)$$

Typically the existential quantifier connects with a conjunction

Translation with quantifiers

- **Assume two predicates S(x) and P(x)**

Universal statements typically tie with implications

- **All S(x) is P(x)**
  - $\forall x\ (S(x) \rightarrow P(x))$
- **No S(x) is P(x)**
  - $\forall x(S(x) \rightarrow \neg P(x))$

Existential statements typically tie with conjunction

- **Some S(x) is P(x)**
  - $\exists x\ (S(x) \wedge P(x))$
- **Some S(x) is not P(x)**
  - $\exists x\ (S(x) \wedge \neg P(x))$

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- There is a person who loves everybody.
- **Translation:**
  - Assume:
    - Variables x and y denote people
    - A predicate L(x,y) denotes: "x loves y"
- Then we can write in the predicate logic:

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- Then we can write in the predicate logic:

  $\exists x \, \forall y \;\; L(x,y)$

Translation exercise

Suppose:

- Variables x,y denote people
- L(x,y) denotes "x loves y".

Translate:

- Everybody loves Raymond. ?

Translation exercise

Suppose:

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Translate:

- Everybody loves Raymond. $\forall x\ L(x,Raymond)$
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- Everybody loves Raymond. $\forall x\ L(x,Raymond)$
- Everybody loves somebody. $\forall x \exists y\ L(x,y)$
- There is somebody whom everybody loves. ?

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Translate:

- Everybody loves Raymond. $\forall x\ L(x,Raymond)$
- Everybody loves somebody. $\forall x \exists y\ L(x,y)$
- There is somebody whom everybody loves. $\exists y \forall x\ L(x,y)$
- There is somebody who Raymond doesn't love. ?

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- There is somebody who Raymond doesn't love.
  $\exists y \neg L(Raymond,y)$
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- There is somebody who Raymond doesn't love.
  $\exists y \neg L(Raymond,y)$
- There is somebody whom no one loves.
  $\exists y\ \forall x\ \neg L(x,y)$

Order of quantifiers

- **Order of quantifiers of the same type does not matter**

  For all x and y, if x is a parent of y then y is a child of x

  $$\forall x, y \ \ parent\ (x, y) \Rightarrow child\ (y, x)$$
  $$\forall y, x \ \ parent\ (x, y) \Rightarrow child\ (y, x)$$

- **Order of different quantifiers changes the meaning**

  $$\forall x \exists y\ loves\ (x, y)$$

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  For all x and y, if x is a parent of y then y is a child of x

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- **Order of different quantifiers changes the meaning**

  $$\forall x \exists y \; loves(x, y)$$

  Everybody loves somebody

  $$\exists y \forall x \; loves(x, y)$$

  There is someone who is loved by everyone

Connections between quantifiers

Everyone likes ice cream

?

Connections between quantifiers

Everyone likes ice cream

$$\forall x \; likes(x, IceCream)$$

Connections between quantifiers

Everyone likes ice cream

$$\forall x\ likes\ (x, IceCream)$$

Is it possible to convey the same meaning using an existential quantifier ?

Connections between quantifiers

Everyone likes ice cream

$$\forall x\ likes\ (x, IceCream)$$

Is it possible to convey the same meaning using an existential quantifier ?

There is no one who does not like ice cream

$$\neg \exists\ x\ \neg\ likes\ (x, IceCream)$$

A universal quantifier in the sentence can be expressed using an existential quantifier !!!

Connections between quantifiers

Someone likes ice cream

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Connections between quantifiers

Someone likes ice cream

$\exists x \; likes(x, IceCream)$

Is it possible to convey the same meaning using a universal quantifier ?

Connections between quantifiers

Someone likes ice cream

$$\exists x \; likes(x, IceCream)$$

Is it possible to convey the same meaning using a universal quantifier ?

Not everyone does not like ice cream

$$\neg \forall x \; \neg likes(x, IceCream)$$

An existential quantifier in the sentence can be expressed using a universal quantifier !!!

Representing knowledge in FOL

Example:

Kinship domain

- **Objects:** people

  $John, Mary, Jane, \ldots$
- **Properties:** gender

  $Male(x), Female(x)$
- **Relations:** parenthood, brotherhood, marriage

  $Parent(x, y), Brother(x, y), Spouse(x, y)$
- **Functions:** mother-of (one for each person x)

  $MotherOf(x)$

Kinship domain in FOL

Relations between predicates and functions: write down what we know about them; how relate to each other.

- Male and female are disjoint categories

$$\forall x\ Male(x) \Leftrightarrow \neg Female(x)$$

- Parent and child relations are inverse

$$\forall x, y\ Parent(x, y) \Leftrightarrow Child(y, x)$$

- A grandparent is a parent of parent

$$\forall g, c\ Grandparent(g, c) \Leftrightarrow \exists p\ Parent(g, p) \wedge Parent(p, c)$$

- A sibling is another child of one's parents

$$\forall x, y\ Sibling(x, y) \Leftrightarrow (x \neq y) \wedge \exists p\ Parent(p, x) \wedge Parent(p, y)$$

- And so on ….