CS 1571 Introduction to AI
Lecture 11

Knowledge Representation. Propositional logic.

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Knowledge-based agent

Knowledge base
Inference engine

- **Knowledge base (KB):**
  - A set of sentences that describe facts about the world in some formal (representational) language
  - **Domain specific**
- **Inference engine:**
  - A set of procedures that use the representational language to infer new facts from known ones or answer a variety of KB queries. Inferences typically require search.
  - **Domain independent**

Example: MYCIN

- MYCIN: an expert system for diagnosis of bacterial infections
- **Knowledge base** represents
  - Facts about a specific patient case
  - Rules describing relations between entities in the bacterial infection domain

If	1. The stain of the organism is gram-positive, and
	2. The morphology of the organism is coccus, and
	3. The growth conformation of the organism is chains
Then	the identity of the organism is streptococcus

- **Inference engine**:
  - manipulates the facts and known relations to answer diagnostic queries (consistent with findings and rules)

Knowledge representation

- The objective of knowledge representation is to express the knowledge about the world in a computer-tractable form

- Key aspects of knowledge representation languages:
  - **Syntax:** describes how sentences are formed in the language
  - **Semantics:** describes the meaning of sentences, what is it the sentence refers to in the real world
  - **Computational aspect:** describes how sentences and objects are manipulated in concordance with semantical conventions

Many KB systems rely on some variant of logic

Logic

A formal language for expressing knowledge and for making logical infereces

Logic is defined by**:**

- **A set of sentences**
  - A sentence is constructed from a set of primitives according to syntax rules.
- **A set of interpretations**
  - An interpretation gives a semantic to primitives. It associates primitives with values.
- **The valuation (meaning) function** V
  - Assigns a value (typically the truth value) to a given sentence under some interpretation

$$V : \text{sentence} \times \text{interpretation} \rightarrow \{True, False\}$$

Propositional logic

- **The simplest logic**
- **Definition:**
  - **A proposition is a statement that is either true or false.**
- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh.
    - **(T)**

Propositional logic

- **The simplest logic**

- **Definition:**
  - **A proposition is a statement that is either true or false.**

- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh.
    - **(T)**
  - $5 + 2 = 8$.
    - **?**

Propositional logic

- **The simplest logic**

- **Definition:**
  - **A proposition is a statement that is either true or false.**

- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh.
    - **(T)**
  - $5 + 2 = 8$.
    - **(F)**
  - It is raining today.
    - **?**

Propositional logic

- The simplest logic

- **Definition:**
  – A proposition is a statement that is either true or false.

- Examples:
  – Pitt is located in the Oakland section of Pittsburgh.
    - (T)
  – 5 + 2 = 8.
    - (F)
  – It is raining today.
    - (either T or F)

Propositional logic

- Examples (cont.):
  – How are you?
    - ?

Propositional logic

- **Examples (cont.):**
  - How are you?
    - **a question is not a proposition**
  - x + 5 = 3
    - **?**

Propositional logic

- **Examples (cont.):**
  - How are you?
    - **a question is not a proposition**
  - x + 5 = 3
    - **since x is not specified, neither true nor false**
  - 2 is a prime number.
    - **?**

Propositional logic

- **Examples (cont.):**
  - How are you?
    - **a question is not a proposition**
  - x + 5 = 3
    - **since x is not specified, neither true nor false**
  - 2 is a prime number.
    - **(T)**
  - She is very talented.
    - ?

Propositional logic

- **Examples (cont.):**
  - How are you?
    - **a question is not a proposition**
  - x + 5 = 3
    - **since x is not specified, neither true nor false**
  - 2 is a prime number.
    - **(T)**
  - She is very talented.
    - **since she is not specified, neither true nor false**
  - There are other life forms on other planets in the universe.
    - **?**

Propositional logic

- **Examples (cont.):**
  - How are you?
    - **a question is not a proposition**
  - $x + 5 = 3$
    - **since x is not specified, neither true nor false**
  - 2 is a prime number.
    - **(T)**
  - She is very talented.
    - **since she is not specified, neither true nor false**
  - There are other life forms on other planets in the universe.
    - **either T or F**

Propositional logic. Syntax

- **Formally propositional logic P:**
  - Is defined **by Syntax+interpretation+semantics of P**

Syntax:

- **Symbols (alphabet)** in P:
  - **Constants**: ***True, False***
  - **Propositional symbols**

    Examples:
    - *P*
    - *Pitt is located in the Oakland section of Pittsburgh.,*
    - *It rains outside,* etc.
  - **A set of connectives**:

$$\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$$

Propositional logic. Syntax

Sentences in the propositional logic:

- **Atomic sentences:**
  - **Constructed from constants and propositional symbols**
  - True, False are (atomic) sentences
  - P, Q or *Light in the room is on, It rains outside* are (atomic) sentences
- **Composite sentences:**
  - **Constructed from valid sentences via connectives**
  - If A, B are sentences then
    $\neg A \quad (A \wedge B) \quad (A \vee B) \quad (A \Rightarrow B) \quad (A \Leftrightarrow B)$
    or $(A \vee B) \wedge (A \vee \neg B)$
    are sentences

Propositional logic. Semantics.

The semantic gives the meaning to sentences.

the semantics in the propositional logic is defined by:

1. **Interpretation of propositional symbols and constants**
   - Semantics of atomic sentences
2. **Through the meaning of connectives**
   - Meaning (semantics) of composite sentences

Semantic: propositional symbols

A **propositional symbol**

- a statement about the world that is either true or false

  Examples:

  - *Pitt is located in the Oakland section of Pittsburgh*
  - *It rains outside*
  - *Light in the room is on*

- An **interpretation** maps symbols to one of the two values: ***True (T),*** or ***False (F)***, depending on whether the symbol is satisfied in the world

  I: *Light in the room is on ->* ***True,*** *It rains outside ->* ***False***

  I': *Light in the room is on ->* ***False,*** *It rains outside ->* ***False***

Semantic: propositional symbols

The **meaning (value)** of the propositional symbol for a specific interpretation is given by its interpretation

I: *Light in the room is on ->* ***True,*** *It rains outside ->* ***False***

V(Light in the room is on, **I***) =* ***True***

V(It rains outside, **I***) =* ***False***

I': *Light in the room is on ->* ***False,*** *It rains outside ->* ***False***

V(Light in the room is on, **I'***) =* ***False***

Semantics: constants

- **The meaning (truth) of constants:**
  - True and False constants are always (under any interpretation) assigned the corresponding ***True,False*** value

$V(True, \mathbf{I}) = \textbf{\textit{True}}$

$V(False, \mathbf{I}) = \textbf{\textit{False}}$

For any interpretation **I**

Semantics: composite sentences.

- **The meaning (truth value) of complex propositional sentences.**
  - Determined using the standard rules of logic:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
True	***True***	***False***	***True***	***True***	***True***	***True***
True	***False***	***False***	***False***	***True***	***False***	***False***
False	***True***	***True***	***False***	***True***	***True***	***False***
False	***False***	***True***	***False***	***False***	***True***	***True***

Translation

Translation of English sentences to propositional logic:

(1) identify atomic sentences that are propositions

(2) Use logical connectives to translate more complex composite sentences that consist of many atomic sentences

Assume the following sentence:

- It is not sunny this afternoon and it is colder than yesterday.

Atomic sentences:

- p = It is sunny this afternoon
- q = it is colder than yesterday

Translation: $\neg p \wedge q$

Translation

Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

Denote:

- p = It is sunny this afternoon
- q = it is colder than yesterday
- r = We will go swimming
- s= we will take a canoe trip
- t= We will be home by sunset

Translation

Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday. $\neg p \wedge q$
- We will go swimming only if it is sunny.
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

Denote:

- p = It is sunny this afternoon
- q = it is colder than yesterday
- r = We will go swimming
- s= we will take a canoe trip
- t= We will be home by sunset

Translation

Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday. $\neg p \wedge q$
- We will go swimming only if it is sunny. $r \rightarrow p$
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

Denote:

- p = It is sunny this afternoon
- q = it is colder than yesterday
- r = We will go swimming
- s= we will take a canoe trip
- t= We will be home by sunset

Translation

Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday. $\neg p \wedge q$
- We will go swimming only if it is sunny. $r \rightarrow p$
- If we do not go swimming then we will take a canoe trip. $\neg r \rightarrow s$
- If we take a canoe trip, then we will be home by sunset.

Denote:

- p = It is sunny this afternoon
- q = it is colder than yesterday
- r = We will go swimming
- s= we will take a canoe trip
- t= We will be home by sunset

Translation

Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday. $\neg p \wedge q$
- We will go swimming only if it is sunny. $r \rightarrow p$
- If we do not go swimming then we will take a canoe trip. $\neg r \rightarrow s$
- If we take a canoe trip, then we will be home by sunset. $s \rightarrow t$

Denote:

- p = It is sunny this afternoon
- q = it is colder than yesterday
- r = We will go swimming
- s= we will take a canoe trip
- t= We will be home by sunset

Contradiction and Tautology

Some composite sentences may always (under any interpretation) evaluate to a single truth value:

- **Contradiction** (always ***False***)

$$P \wedge \neg P$$

- **Tautology (**always ***True*****)**

$$P \vee \neg P$$

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$
$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

} DeMorgan's Laws

Model, validity and satisfiability

- A **model (in logic):** An interpretation is a model for a set of sentences if it assigns true to each sentence in the set.
- A sentence is **satisfiable** if it has a model;
  - There is at least one interpretation under which the sentence can evaluate to True.
- A sentence is **valid** if it is ***True*** in all interpretations
  - i.e., if its negation is **not satisfiable** (leads to contradiction)

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	***True***	***True***	***False***	***True***
True	***False***	***True***	***True***	***True***
False	***True***	***True***	***False***	***True***
False	***False***	***False***	***False***	***True***

Model, validity and satisfiability

- A **model (in logic):** An interpretation is a model for a set of sentences if it assigns true to each sentence in the set.
- A sentence is **satisfiable** if it has a model;
  - There is at least one interpretation under which the sentence can evaluate to True.
- A sentence is **valid** if it is ***True*** in all interpretations
  - i.e., if its negation is **not satisfiable** (leads to contradiction)

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	***True***	***True***	***False***	***True***
True	***False***	***True***	***True***	***True***
False	***True***	***True***	***False***	***True***
False	***False***	***False***	***False***	***True***

Model, validity and satisfiability

- A **model (in logic):** An interpretation is a model for a set of sentences if it assigns true to each sentence in the set.
- A sentence is **satisfiable** if it has a model;
  - There is at least one interpretation under which the sentence can evaluate to True.
- A sentence is **valid** if it is ***True*** in all interpretations
  - i.e., if its negation is **not satisfiable** (leads to contradiction)

Satisfiable sentence

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	***True***	***True***	***False***	***True***
True	***False***	***True***	***True***	***True***
False	***True***	***True***	***False***	***True***
False	***False***	***False***	***False***	***True***

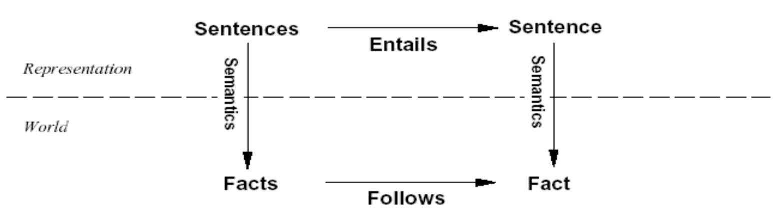
Model, validity and satisfiability

- A **model (in logic):** An interpretation is a model for a set of sentences if it assigns true to each sentence in the set.
- A sentence is **satisfiable** if it has a model;
  - There is at least one interpretation under which the sentence can evaluate to True.
- A sentence is **valid** if it is ***True*** in all interpretations
  - i.e., if its negation is **not satisfiable** (leads to contradiction)

			Satisfiable sentence	Valid sentence
P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	***True***	***True***	***False***	***True***
True	***False***	***True***	***True***	***True***
False	***True***	***True***	***False***	***True***
False	***False***	***False***	***False***	***True***

Entailment

- **Entailment** reflects the relation of one fact in the world following from the others



- Entailment $KB \models \alpha$
- Knowledge base KB entails sentence α if and only if α is true in all worlds where KB is true

Sound and complete inference.

Inference is a process by which conclusions are reached.

- We want to implement the inference process on a computer !!

Assume an **inference procedure *i*** that

- derives a sentence α from the KB : $KB \vdash_i \alpha$

Properties of the inference procedure in terms of entailment

- **Soundness:** An inference procedure is **sound**

  If $KB \vdash_i \alpha$ then it is true that $KB \models \alpha$

- **Completeness:** An inference procedure is **complete**

  If $KB \models \alpha$ then it is true that $KB \vdash_i \alpha$

Logical inference problem

Logical inference problem:

- **Given:**
  - a knowledge base KB (a set of sentences) and
  - a sentence α (called **a theorem**),
- **Does a KB semantically entail** α ? $KB \models \alpha$?

In other words: In all interpretations in which sentences in the KB are true, is also α true?

Question: Is there a procedure (program) that can decide this problem in a finite number of steps?

Answer: **Yes**. Logical inference problem for the propositional logic is **decidable**.

Solving logical inference problem

In the following:

How to design the procedure that answers:

$$KB \models \alpha \ ?$$

Three approaches:

- **Truth-table approach**
- **Inference rules**
- **Conversion to the inverse SAT problem**
  - **Resolution-refutation**

Truth-table approach

Problem: $KB \models \alpha \ ?$

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False values to propositional symbols)

Example:

		KB		α
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$
True	*True*	*True*	*True*	*True*
True	*False*	*True*	*False*	*False*
False	*True*	*True*	*False*	*False*
False	*False*	*False*	*True*	*False*

Truth-table approach

Problem: $KB \models \alpha$?

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

Example:

		KB		α
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$
True	*True*	*True*	*True*	*True*
True	*False*	*True*	*False*	*False*
False	*True*	*True*	*False*	*False*
False	*False*	*False*	*True*	*False*

Truth-table approach

Problem: $KB \models \alpha$?

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

Example:

		KB		α	
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$	
True	*True*	*True*	*True*	*True*	✓
True	*False*	*True*	*False*	*False*	
False	*True*	*True*	*False*	*False*	
False	*False*	*False*	*True*	*False*	

Truth-table approach

A two steps procedure:

1. **Generate table for all possible interpretations**
2. Check whether the sentence α evaluates to true whenever KB evaluates to true

Example: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	*True*	*True*				
True	*True*	*False*				
True	*False*	*True*				
True	*False*	*False*				
False	*True*	*True*				
False	*True*	*False*				
False	*False*	*True*				
False	*False*	*False*				

Truth-table approach

A two steps procedure:

1. Generate table for all possible interpretations
2. **Check whether the sentence α evaluates to true whenever *KB* evaluates to true**

Example: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	*True*	*True*	*True*	*True*	*True*	*True*
True	*True*	*False*	*True*	*True*	*True*	*True*
True	*False*	*True*	*True*	*False*	*False*	*True*
True	*False*	*False*	*True*	*True*	*True*	*True*
False	*True*	*True*	*True*	*True*	*True*	*True*
False	*True*	*False*	*False*	*True*	*False*	*True*
False	*False*	*True*	*True*	*False*	*False*	*False*
False	*False*	*False*	*False*	*True*	*False*	*False*

Truth-table approach

A two steps procedure:

1. Generate table for all possible interpretations
2. **Check whether the sentence α evaluates to true whenever *KB* evaluates to true**

Example: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α	
True	True	True	True	True	True	True	✓
True	True	False	True	True	True	True	✓
True	False	True	True	False	False	True	
True	False	False	True	True	True	True	✓
False	True	True	True	True	True	True	✓
False	True	False	False	True	False	True	
False	False	True	True	False	False	False	
False	False	False	False	True	False	False	

Truth-table approach

$KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	True	True	True	True	True	True
True	True	False	True	True	True	True
True	False	True	True	False	False	True
True	False	False	True	True	True	True
False	True	True	True	True	True	True
False	True	False	False	True	False	True
False	False	True	True	False	False	False
False	False	False	False	True	False	False

KB entails α

- The **truth-table approach** is **sound and complete** for the propositional logic!!