CS 1571 Introduction to AI
Lecture 8

Constraint satisfaction search

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Constraint satisfaction problem (CSP)

Objective:

- **Find a configuration** satisfying goal conditions

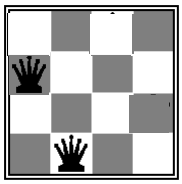
- **Constraint satisfaction problem (CSP)** is **a configuration search problem** where**:**
  - A **state** is defined by **a set of variables and their values**
  - **Goal condition** is represented by **a set constraints on possible variable values**

CSP example: N-queens

Goal: n queens placed in non-attacking positions on the board

Variables:

- Represent queens, one for each column:
  - Q_1, Q_2, Q_3, Q_4
- Values:
  - Row placement of each queen on the board
    {1, 2, 3, 4}



$Q_1 = 2, Q_2 = 4$

Constraints: $Q_i \neq Q_j$ Two queens not in the same row

$|Q_i - Q_j| \neq |i - j|$ Two queens not on the same diagonal

CSP example: Satisfiability (SAT) problem

Determine whether a sentence in the conjunctive normal form (CNF) is satisfiable (can evaluate to true)

- Used in the propositional logic (covered later)

$$(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg R \vee S) \wedge (\neg P \vee Q \vee \neg T) \ldots$$

Variables:

- Propositional symbols (P, R, T, S)
- Values: *True, False*

Constraints:

- Every conjunct must evaluate to true, at least one of the literals must evaluate to true

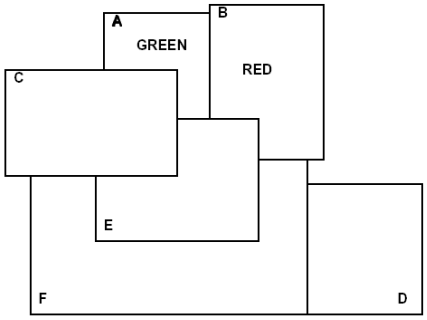
$$(P \vee Q \vee \neg R) \equiv True, (\neg P \vee \neg R \vee S) \equiv True, \ldots$$

CSP example: Map coloring

Color a map using k different colors such that no adjacent countries have the same color

Variables: ?

- Variable values: ?



Constraints: ?

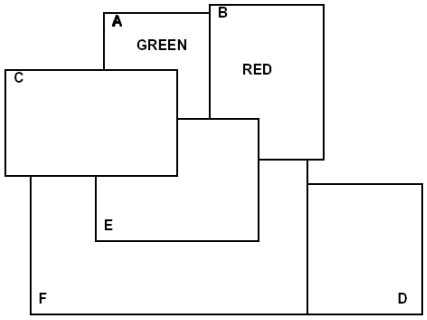
Map coloring

Color a map using k different colors such that no adjacent countries have the same color

Variables:

- Represent countries
  - A, B, C, D, E
- Values:
  - K -different colors

  {Red, Blue, Green,..}



Constraints: ?

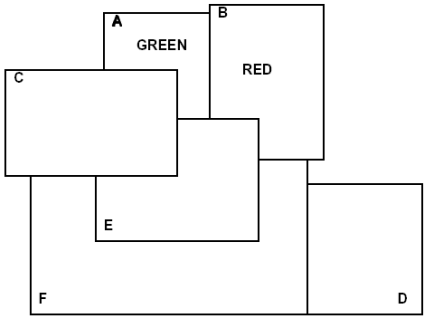
Map coloring

Color a map using k different colors such that no adjacent countries have the same color

Variables:

- Represent countries
  - A, B, C, D, E
- Values:
  - K -different colors

  {Red, Blue, Green,..}



Constraints: $A \neq B, A \neq C, C \neq E$, etc

An example of a problem with **binary constraints**

Constraint satisfaction as a search problem

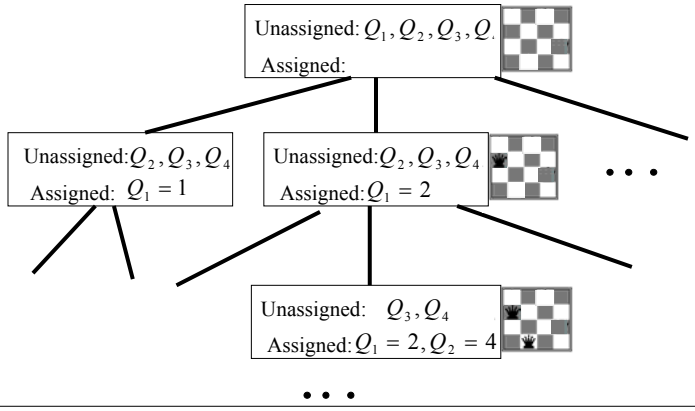
Formulation of a CSP as a search problem:

- **States.** Assignment (partial, complete) of values to variables.
- **Initial state.** No variable is assigned a value.
- **Operators.** Assign a value to one of the unassigned variables.
- **Goal condition.** All variables are assigned, no constraints are violated.

- **Constraints** can be **represented**:
  - **Explicitly** by a set of allowable values
  - **Implicitly** by a function that tests for the satisfaction of constraints

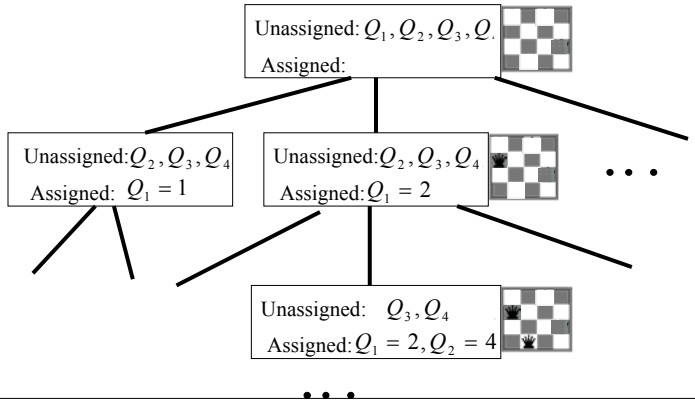
Solving a CSP through standard search

- **Maximum depth of the tree (m): ?**
- **Depth of the solution (d) : ?**
- **Branching factor (b) : ?**



Solving a CSP through standard search

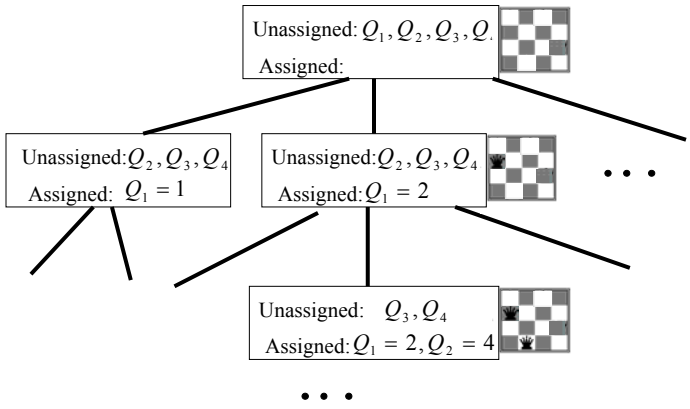
- **Maximum depth of the tree:** Number of variables in the CSP
- **Depth of the solution:** Number of variables in the CSP
- **Branching factor:** if we fix the order of variable assignments the branch factor depends on the number of their values



Solving a CSP through standard search

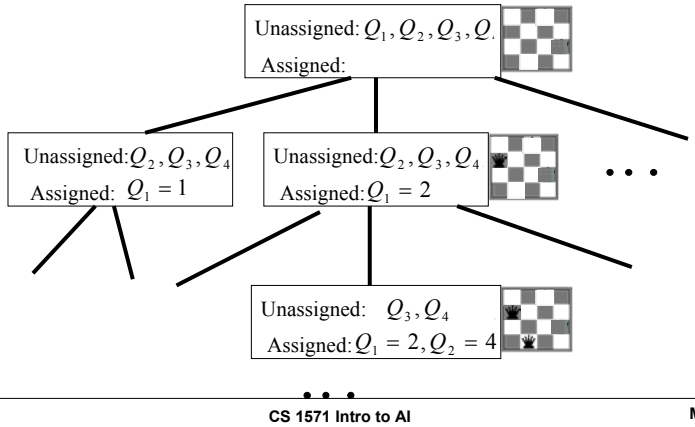
- **What search algorithm to use: ?**

  Depth of the tree = Depth of the solution=number of vars



Solving a CSP through standard search

- **What search algorithm to use: Depth first search !!!**
  - Since we know the depth of the solution
  - We do not have to keep large number of nodes in queues



Backtracking

Depth-first search for CSP is also referred to as **backtracking**

The violation of constraints needs to be checked for each node**,** either during its generation or before its expansion

Consistency of constraints:

- Current **variable assignments** together **with constraints restrict remaining legal values of unassigned variables**;
- The remaining **legal and illegal values of variables may be inferred** (effect of constraints propagates)
- To prevent "blind" exploration it is necessary to keep track of the remaining legal values, so we know when the constraints are violated and when to terminate the search

Constraint propagation

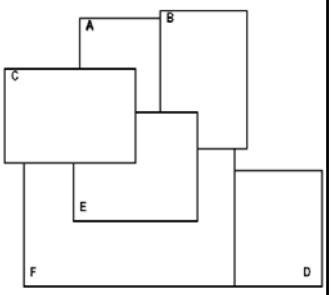
A **state** (more broadly) is defined by a set of variables, their values and a list of legal and illegal assignments for unassigned variables

Legal and illegal assignments can be represented via: **equations** (value assignments) and **disequations (list of invalid assignments)**

Example: map coloring

Equation $A = \text{Red}$

Disequation $C \neq \text{Red}$



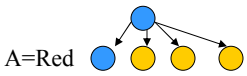
Constraints + assignments
can entail new equations and disequations

$A = \text{Red} \rightarrow B \neq \text{Red}$

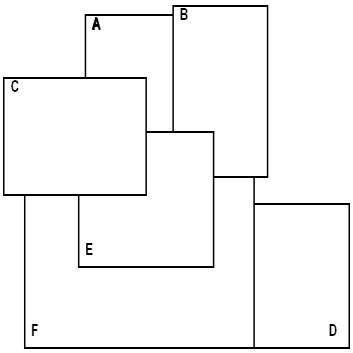
Constraint propagation: the process of inferring of new equations and disequations from existing equations and disequations

Constraint propagation

- Assign A=Red



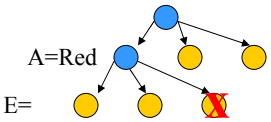
	Red	Blue	Green
A	✓		
B			
C			
D			
E			
F			



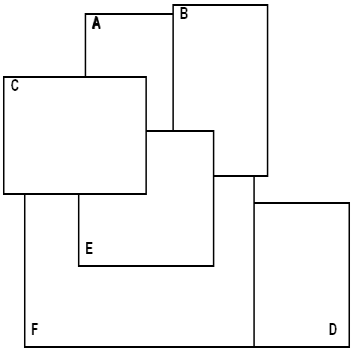
✓ - equations ✗ - disequations

Constraint propagation

- Assign A=Red



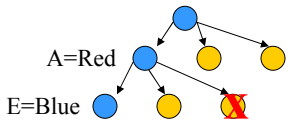
	Red	Blue	Green
A	✓		
B	✗		
C	✗		
D			
E	✗		
F			



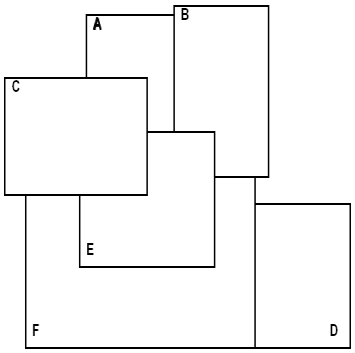
✓ - equations ✗ - disequations

Constraint propagation

- Assign E=Blue

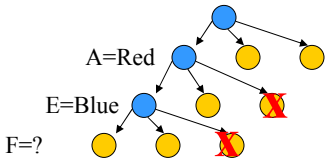


	Red	Blue	Green
A	✓		
B	✗		
C	✗		
D			
E	✗	✓	
F			

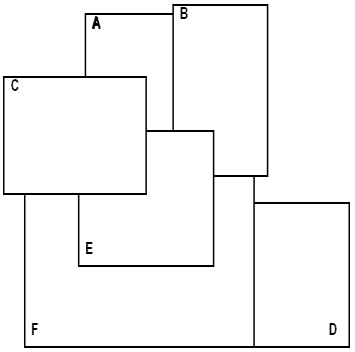


Constraint propagation

- Assign E=Blue



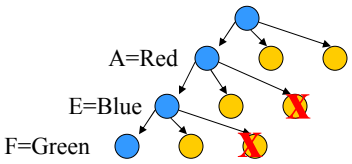
	Red	Blue	Green
A	✓	✗	
B	✗	✗	
C	✗	✗	
D			
E	✗	✓	
F		✗	

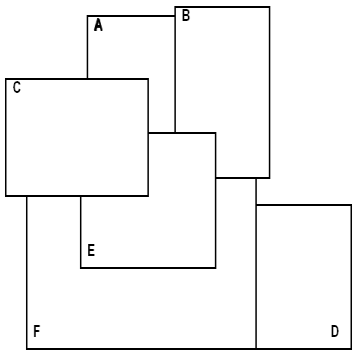


Constraint propagation

- Assign F=Green

	Red	Blue	Green
A	✓	✗	
B	✗	✗	
C	✗	✗	
D			
E	✗	✓	
F		✗	✓

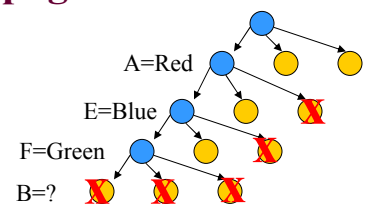


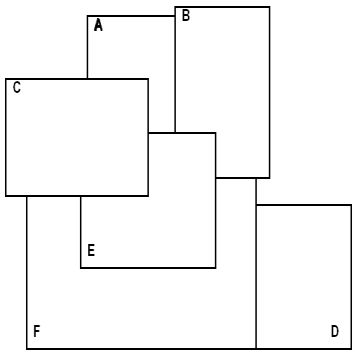


Constraint propagation

- Assign F=Green

	Red	Blue	Green
A	✓	✗	
B	✗	✗	✗
C	✗	✗	✗
D			✗
E	✗	✓	✗
F		✗	✓

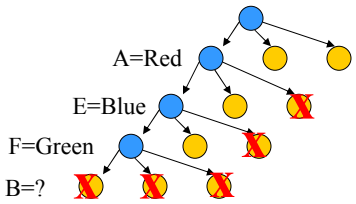


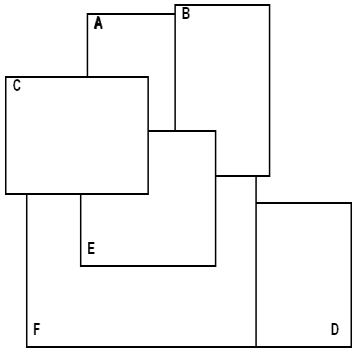


Constraint propagation

- Assign F=Green

	Red	Blue	Green
A	✓	✗	
B	✗	✗	✗
C	✗	✗	✗
D			✗
E	✗	✓	✗
F		✗	✓



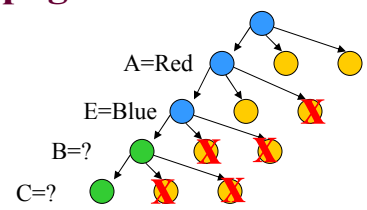


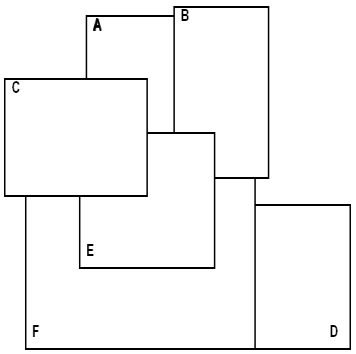
Conflict !!! No legal assignments available for B and C

Constraint propagation

- We can derive remaining legal values through propagation

	Red	Blue	Green
A	✓	✗	
B	✗	✗	✓
C	✗	✗	✓
D			
E	✗	✓	
F		✗	





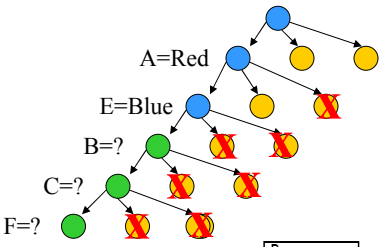
B=Green

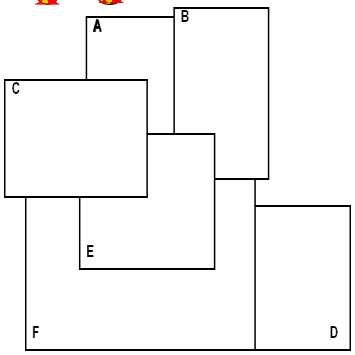
C=Green

Constraint propagation

- We can derive remaining legal values through propagation

	Red	Blue	Green
A	✓	✗	✗
B	✗	✗	✓
C	✗	✗	✓
D	✗		
E	✗	✓	✗
F	✓	✗	✗



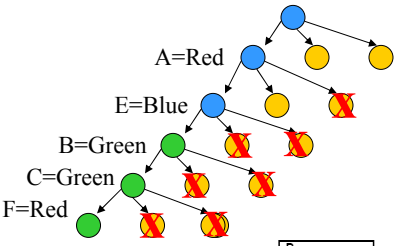


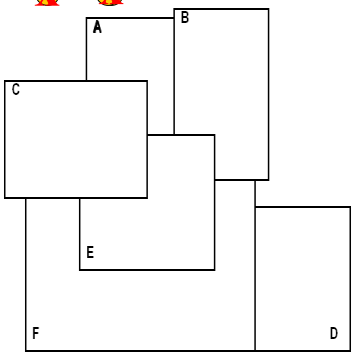
B=Green
C=Green ⇒ **F=Red**

Constraint propagation

- We can derive remaining legal values through propagation

	Red	Blue	Green
A	✓	✗	✗
B	✗	✗	✓
C	✗	✗	✓
D	✗		
E	✗	✓	✗
F	✓	✗	✗





B=Green
C=Green ⇒ **F=Red**

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Constraint propagation

Three known techniques for propagating the effects of past assignments and constraints:

- **Value propagation**
- **Arc consistency**
- **Forward checking**

- **Difference:**
  - Completeness of inferences
  - Time complexity of inferences.

Constraint propagation

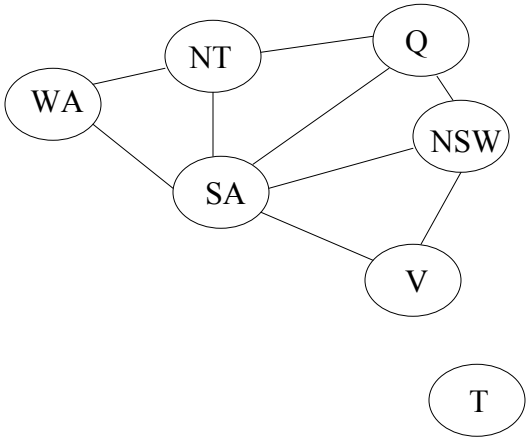
1. **Value propagation. Infers:**
   - **equations from** the set of **equations** defining the partial assignment, **and a constraint**
2. **Arc consistency. Infers:**
   - **disequations from** the set of **equations and disequations** defining the partial assignment, and **a constraint**
   - **equations through the exhaustion of alternatives**
3. **Forward checking. Infers:**
   - **disequations from** a set of **equations** defining the partial assignment, and a constraint
   - **Equations through the exhaustion of alternatives**

Restricted forward checking:

- uses only active constraints (active constraint – only one variable unassigned in the constraint)

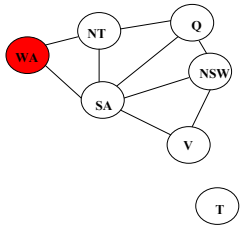
Example

Map coloring of Australia territories



Example: forward checking

Map coloring

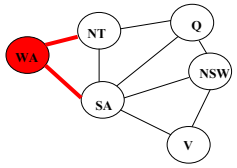


Set: WA=Red

vars	WA	NT	Q	NSW	V	SA	T
domain	R G B	R G B	R G B	R G B	R G B	R G B	R G B
WA=Red	**R**	?	?	?	?	?	?

Example: forward checking

Map coloring

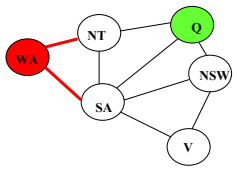


Set: WA=Red

vars	WA	NT	Q	NSW	V	SA	T
domain	R G B	R G B	R G B	R G B	R G B	R G B	R G B
WA=Red	**R**	G B	R G B	R G B	R G B	G B	R G B

Example: forward checking

Map coloring

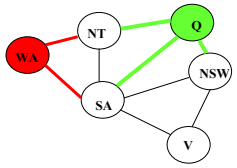


Set: Q=Green

vars	WA	NT	Q	NSW	V	SA	T
domain	R G B	R G B	R G B	R G B	R G B	R G B	R G B
WA=Red	**R**	G B	R G B	R G B	R G B	G B	R G B
Q=Green	**R**	?	**G**	?	?	?	?

Example: forward checking

Map coloring



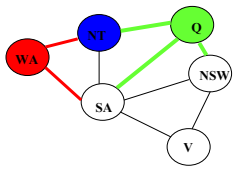
Set: Q=Green

T

vars	WA	NT	Q	NSW	V	SA	T
domain	R G B	R G B	R G B	R G B	R G B	R G B	R G B
WA=Red	**R**	G B	R G B	R G B	R G B	G B	R G B
Q=Green	**R**	B	**G**	R B	R G B	B	R G B

Example: forward checking

Map coloring



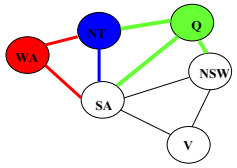
Infer: **Exhaustions of alternatives**

T

vars	WA	NT	Q	NSW	V	SA	T
domain	R G B	R G B	R G B	R G B	R G B	R G B	R G B
WA=Red	**R**	G B	R G B	R G B	R G B	G B	R G B
Q=Green	**R**	B	**G**	R B	R G B	B	R G B
Infer NT	**R**	**B**	**G**	?	?	?	?

Example: forward checking

Map coloring

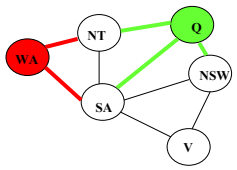


Infer: **Exhaustions of alternatives**

vars	WA	NT	Q	NSW	V	SA	T
domain	R G B	R G B	R G B	R G B	R G B	R G B	R G B
WA=Red	**R**	G B	R G B	R G B	R G B	G B	R G B
Q=Green	**R**	B	**G**	R B	R G B	B	R G B
Infer NT	**R**	**B**	**G**	R B	R G B	!	R G B

Example: arc consistency

Map coloring

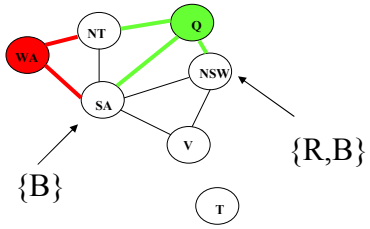


Set: WA=Red
Set: Q=Green

vars	WA	NT	Q	NSW	V	SA	T
domain	R G B	R G B	R G B	R G B	R G B	R G B	R G B
WA=Red	**R**	G B	R G B	R G B	R G B	G B	R G B
Q=Green	**R**	B	**G**	R B	R G B	B	R G B

Example: arc consistency

Map coloring

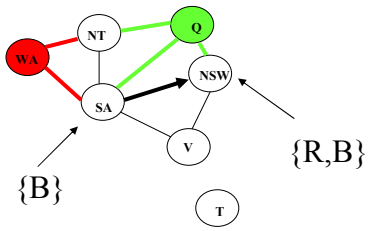


Set: WA=Red
Set: Q=Green

vars	WA	NT	Q	NSW	V	SA	T
domain	R G B	R G B	R G B	R G B	R G B	R G B	R G B
WA=Red	**R**	G B	R G B	R G B	R G B	G B	R G B
Q=Green	**R**	B	**G**	R B	R G B	B	R G B

Example: arc consistency

Map coloring



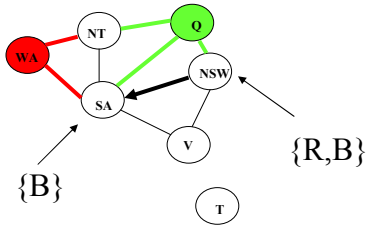
SA=B
NSW=R
Consistent assignment

Set: WA=Red
Set: Q=Green

vars	WA	NT	Q	NSW	V	SA	T
domain	R G B	R G B	R G B	R G B	R G B	R G B	R G B
WA=Red	**R**	G B	R G B	R G B	R G B	G B	R G B
Q=Green	**R**	B	**G**	R B	R G B	B	R G B

Example: arc consistency

Map coloring



Set: WA=Red
Set: Q=Green

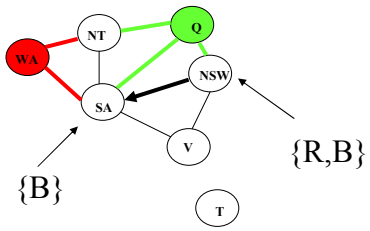
NSW=B
SA=!

Inconsistent assignment

vars	WA	NT	Q	NSW	V	SA	T
domain	R G B	R G B	R G B	R G B	R G B	R G B	R G B
WA=Red	**R**	G B	R G B	R G B	R G B	G B	R G B
Q=Green	**R**	B	**G**	R B	R G B	B	R G B

Example: arc consistency

Map coloring



Set: WA=Red
Set: Q=Green

NSW=B
SA=!

Inconsistent assignment

vars	WA	NT	Q	NSW	V	SA	T
domain	R G B	R G B	R G B	R G B	R G B	R G B	R G B
WA=Red	**R**	G B	R G B	R G B	R G B	G B	R G B
Q=Green	**R**	B	**G**	R B	R G B	B	R G B
				R			

Heuristics for CSPs

CSP searches the space in the depth-first manner.

But we still can choose:

- **Which variable to assign next?**
- **Which value to choose first?**

Heuristics

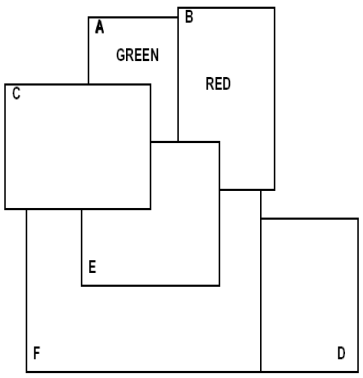
- **Most constrained variable**
  - Which variable is likely to become a bottleneck?
- **Least constraining value**
  - Which value gives us more flexibility later?

Heuristics for CSP

Examples: **map coloring**

Heuristics

- **Most constrained variable**
  - ?
- **Least constraining value**
  - ?

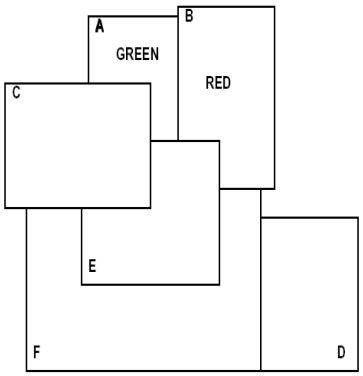


Heuristics for CSP

Examples: **map coloring**

Heuristics

- **Most constrained variable**
  - Country E is the most constrained one (cannot use Red, Green)
- **Least constraining value**
  - ?

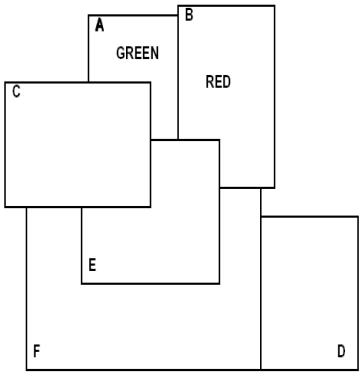


Heuristics for CSP

Examples: **map coloring**

Heuristics

- **Most constrained variable**
  - Country E is the most constrained one (cannot use Red, Green)
- **Least constraining value**
  - Assume we have chosen variable C
  - What color is the least constraining color?

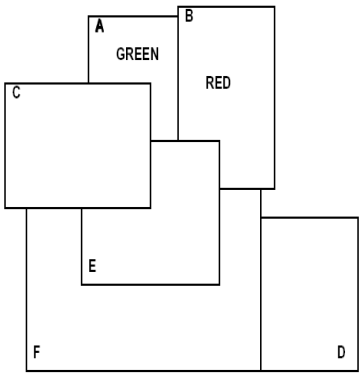


Heuristics for CSP

Examples: **map coloring**

Heuristics

- **Most constrained variable**
  - Country E is the most constrained one (cannot use Red, Green)
- **Least constraining value**
  - Assume we have chosen variable C
  - Red is the least constraining valid color for the future



Finding optimal configurations

Search for the optimal configuration

Constrain satisfaction problem:

Objective: find a configuration that satisfies all constraints

Optimal configuration problem:

Objective: find the best configuration

The quality of a configuration: is defined by some quality measure that reflects our **preference towards each configuration** (or state)

Our goal: optimize the configuration according to the quality measure also referred to as objective function

Search for the optimal configuration

If the space of configurations we search among is

- **Discrete or finite**
  - **then it is a combinatorial optimization problem**
- **Continuous**
  - **then it is a parametric optimization problem**

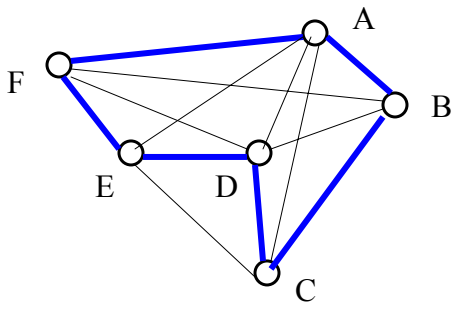
In the following we cover with combinatorial optimization problems

Parametric optimization will covered next lecture.

Example: Traveling salesman problem

Problem:

- A graph with distances

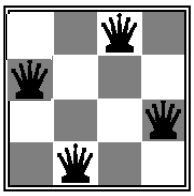


- **Goal:** find the shortest tour which visits every city once and returns to the start

  An example of a valid **tour:** ABCDEF

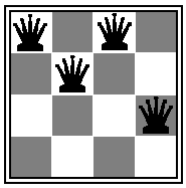
Example: N queens

- A CSP problem
- Is it possible to formulate the problem as an optimal configuration search problem ?

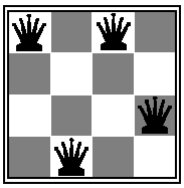


Example: N queens

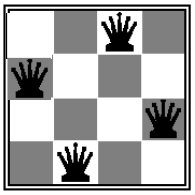
- A CSP problem
- Is it possible to formulate the problem as an optimal configuration search problem ? **Yes.**
- **The quality of a configuration in a CSP** can be measured by the number of violated constraints
- **Solving:** minimize the number of constraint violations



# of violations =3



# of violations =1



# of violations =0

Iterative optimization methods

- Searching systematically for the best configuration with the **DFS** may not be the best solution
- Worst case running time:
  - Exponential in the number of variables
- Solutions to **large 'optimal' configuration** problems are often found using iterative optimization methods

- **Methods:**
  - **Hill climbing**
  - **Simulated Annealing**
  - **Genetic algorithms**