CS 1571 Introduction to AI
Lecture 20

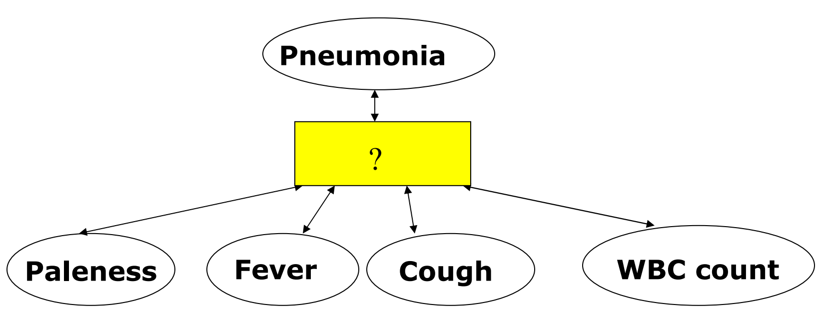
Modeling uncertainty

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Modeling the uncertainty.

Key challenges:

- How to represent nondeterministic (stochastic) relations?
- How to manipulate such knowledge to make inferences?
  - **Humans can reason with uncertainty.**



Methods for representing uncertainty

Probability theory

- A well defined theory for modeling and reasoning in the presence of uncertainty
- A natural choice to replace certainty factors

Facts (propositional statements)

- Are represented via **random variables** with two or more values

  Example: *Pneumonia* is a random variable

  values: *True* and *False*

- Each value can be achieved **with some probability:**

$$P(Pneumonia = True) = 0.001$$

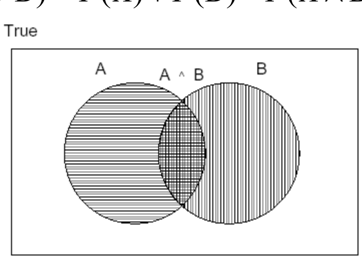
$$P(WBCcount = high) = 0.005$$

Probability theory

- Well-defined theory for representing and manipulating statements with uncertainty
- **Axioms of probability:**

  For any two propositions A, B.

  1. $0 \le P(A) \le 1$
  2. $P(True) = 1$ and $P(False) = 0$
  3. $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$



Modeling uncertainty with probabilities

Probabilistic extension of propositional logic.

- **Propositions:**
  - statements about the world
  - Represented by the assignment of values to **random variables**
- **Random variables:**
  - ! **Boolean** $\underbrace{Pneumonia}_{\text{Random variable}}$ is either $\underbrace{True, False}_{\text{Values}}$
  - ! **Multi-valued** $\underbrace{Pain}_{\text{Random variable}}$ is one of $\underbrace{\{Nopain, Mild, Moderate, Severe\}}_{\text{Values}}$
  - **Continuous** $\underbrace{HeartRate}_{\text{Random variable}}$ is a value in $\underbrace{<0;250>}_{\text{Values}}$

Probabilities

Unconditional probabilities

$P(Pneumonia) = 0.001$ or $P(Pneumonia = True) = 0.001$

$P(Pneumonia = False) = 0.999$

$P(WBCcount = high) = 0.005$

Probability distribution

- Defines probabilities **for all possible value assignments to a random variable**
- Values are mutually exclusive

$P(Pneumonia = True) = 0.001$
$P(Pneumonia = False) = 0.999$

$Pneumonia$	$\mathbf{P}(Pneumonia)$
$True$	0.001
$False$	0.999

Probability distribution

Defines probability for **all possible value assignments**

Example 1:

$P(Pneumonia = True) = 0.001$

$P(Pneumonia = False) = 0.999$

$Pneumonia$	$\mathbf{P}(Pneumonia)$
$True$	0.001
$False$	0.999

$$P(Pneumonia = True) + P(Pneumonia = False) = 1$$

Probabilities sum to 1 !!!

Example 2:

$P(WBCcount = high) = 0.005$

$P(WBCcount = normal) = 0.993$

$P(WBCcount = high) = 0.002$

$WBCcount$	$\mathbf{P}(WBCcount)$
$high$	0.005
$normal$	0.993
low	0.002

Joint probability distribution

Joint probability distribution (for a set variables)

- Defines probabilities for **all possible assignments of values to variables in the set**

Example: variables *Pneumonia* and *WBCcount*

$\mathbf{P}(pneumonia, WBCcount)$

Is represented by 2×3 matrix

		$WBCcount$		
		$high$	$normal$	low
$Pneumonia$	$True$	0.0008	0.0001	0.0001
	$False$	0.0042	0.9929	0.0019

Joint probabilities

Marginalization

- reduces the dimension of the joint distribution
- Sums variables out

$\mathbf{P}(pneumonia, WBCcount)$ 2×3 matrix

		WBCcount			
		high	*normal*	*low*	$\mathbf{P}(Pneumonia)$
Pneumonia	*True*	0.0008	0.0001	0.0001	0.001
	False	0.0042	0.9929	0.0019	0.999
$\mathbf{P}(WBCcount)$		0.005	0.993	0.002	

Marginalization (here summing of columns or rows)

Marginalization

Marginalization

- reduces the dimension of the joint distribution

$$P(X_1, X_2, \ldots X_{n-1}) = \sum_{\{X_n\}} P(X_1, X_2, \ldots X_{n-1}, X_n)$$

- We can continue doing this

$$P(X_2, \ldots X_{n-1}) = \sum_{\{X_1, X_n\}} P(X_1, X_2, \ldots X_{n-1}, X_n)$$

What is the maximal joint probability distribution?

- **Full joint probability**

Full joint distribution

- **the joint distribution for all variables in the problem**
  - It defines the complete probability model for the problem

Example: pneumonia diagnosis

Variables: *Pneumonia, Fever, Paleness, WBCcount, Cough*

Full joint defines the probability for all possible assignments of values to *Pneumonia, Fever, Paleness, WBCcount, Cough*

5 variables: full joint is a captured by a 5-dimensional table

$$P(Pneumonia = T, WBCcount = High, Fever = T, Cough = T, Paleness = T)$$

$$P(Pneumonia = T, WBCcount = High, Fever = T, Cough = T, Paleness = F)$$

$$P(Pneumonia = T, WBCcount = High, Fever = T, Cough = F, Paleness = T)$$

… etc

Full joint distribution

- **Any joint probability for a subset of variables can be obtained from the full joint via marginalization**

$$P(Pneumonia, WBCcount, Fever) = \sum_{c,p=\{T,F\}} P(Pneumonia, WBCcount, Fever, Cough = c, Paleness = p)$$

- **Is it possible to recover full joint from the joint probabilities over a subset of variables?**

Full joint distribution

- **Any joint probability for a subset of variables can be obtained from the full joint via marginalization**

$$P(Pneumonia, WBCcount, Fever) =$$
$$\sum_{c,p=\{T,F\}} P(Pneumonia, WBCcount, Fever, Cough = c, Paleness = p)$$

- **Is it possible to recover the joint distribution for a set of variables from joint probabilities defined for its subsets?**

Relations among joint distributions

Assume:

	True	**False**
P(Pneumonia)	0.001	0.999

	True	**False**
P(Fever)	0.05	0.95

Can we unambiguously compute the joint over the two variables?

P(Fever, Pneumonia)	**True**	**False**
True	?	?
False	?	?

Relations among joint distributions

Assume:

	True	**False**
P(Pneumonia)	0.001	0.999

	True	**False**
P(Fever)	0.05	0.95

Can we unambiguously compute the joint over the two variables?

No! More than one probability value is possible for joint table entries

P(Fever, Pneumonia)

	True	**False**
True	?	?
False	?	?

Relations among joint distribution

Assume:

	True	**False**	
P(Pneumonia)	0.001	0.999	← 1 free parameter

	True	**False**	
P(Fever)	0.05	0.95	← 1 free parameter

Can we unambiguously compute the joint over the two variables?

The joint has more free parameters, the two individual distributions together

P(Fever, Pneumonia)

	True	**False**
True	?	?
False	?	?

← 3 free parameters

Relations among joint distribution

Assume:

	True	False	
P(Pneumonia)	0.001	0.999	← 1 free parameter

	True	False	
P(Fever)	0.05	0.95	← 1 free parameter

Is there a condition that would let us unambiguously compute the joint over two variables?

Yes. When the two variables are independent!

P(Fever, Pneumonia)

	True	False
True	?	?
False	?	?

← 3 free parameters

Relations among joint distribution

Assume:

	True	False	
P(Pneumonia)	0.001	0.999	← 1 free parameter

	True	False	
P(Fever)	0.05	0.95	← 1 free parameter

P(Pneumonia=True)*P(Fever=True)

P(Fever, Pneumonia)

	True	False
True	0.00005	0.0495
False	0.0095	?

← 3 free parameters

Conditional probabilities

Conditional probability distribution

- Defines probabilities for all possible assignments, given a fixed assignment to some other variable values

$P(Pneumonia = true \mid WBCcount = high)$

$\mathbf{P}(Pneumonia \mid WBCcount)$ 3 element vector of 2 elements

		WBCcount		
		high	*normal*	*low*
Pneumonia	*True*	0.08	0.0001	0.0001
	False	0.92	0.9999	0.9999
		1.0	1.0	1.0

$P(Pneumonia = true \mid WBCcount = high)$
$+ P(Pneumonia = false \mid WBCcount = high)$

Conditional probabilities

Conditional probability

- Is defined in terms of the joint probability:

$$P(A \mid B) = \frac{P(A,B)}{P(B)} \text{ s.t. } P(B) \neq 0$$

- **Example:**

$$P(pneumonia = true \mid WBCcount = high) = \frac{P(pneumonia = true, WBCcount = high)}{P(WBCcount = high)}$$

$$P(pneumonia = false \mid WBCcount = high) = \frac{P(pneumonia = false, WBCcount = high)}{P(WBCcount = high)}$$

Conditional probabilities

- **Conditional probability distribution.**

$$P(A \mid B) = \frac{P(A,B)}{P(B)} \quad \text{s.t. } P(B) \neq 0$$

- **Product rule.** Join probability can be expressed in terms of conditional probabilities

$$P(A,B) = P(A \mid B)P(B)$$

- **Chain rule**. Any joint probability can be expressed as a product of conditionals

$$P(X_1, X_2, \ldots X_n) = P(X_n \mid X_{1,} \ldots X_{n-1})P(X_{1,} \ldots X_{n-1})$$
$$= P(X_n \mid X_{1,} \ldots X_{n-1})P(X_{n-1} \mid X_{1,} \ldots X_{n-2})P(X_{1,} \ldots X_{n-2})$$
$$= \prod_{i=1}^{n} P(X_i \mid X_{1,} \ldots X_{i-1})$$

Bayes rule

Conditional probability.

$$P(A \mid B) = \frac{P(A,B)}{P(B)} \qquad \longrightarrow \qquad P(A,B) = P(B \mid A)P(A)$$

Bayes rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

When is it useful?

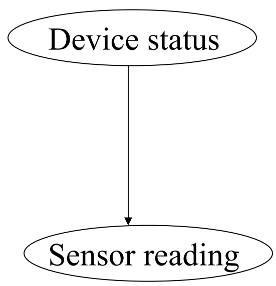
- When we are interested in computing the diagnostic query from the causal probability

$$P(cause \mid effect) = \frac{P(effect \mid cause)P(cause)}{P(effect)}$$

- **Reason:** It is often easier to assess causal probability
  - E.g. Probability of pneumonia causing fever
    vs. probability of pneumonia given fever

Bayes Rule in a simple diagnostic inference

- **Device** (equipment) operating *normally* or *malfunctioning*.
  - Operation of the device sensed indirectly via a sensor
- **Sensor reading** is either *high* or *low*



P(Device status)

normal	**malfunctioning**
0.9	0.1

P(Sensor reading| Device status)

Device\Sensor	**high**	**low**
normal	0.1	0.9
malfunctioning	0.6	0.4

Bayes Rule in a simple diagnostic inference.

- **Diagnostic inference:** compute the probability of device operating normally or malfunctioning given a sensor reading

$$\mathbf{P}(\text{Device status} \mid \text{Sensor reading} = high) = ?$$

$$= \begin{pmatrix} P(\text{Device status} = normal \mid \text{Sensor reading} = high) \\ P(\text{Device status} = malfunctioning \mid \text{Sensor reading} = high) \end{pmatrix}$$

- Note that typically the opposite conditional probabilities are given to us: they are much easier to estimate
- **Solution:** apply **Bayes rule** to reverse the conditioning variables

Bayes rule

Assume a variable A with multiple values $a_1, a_2, \ldots a_k$

Bayes rule can be rewritten as:

$$P(A = a_j \mid B = b) = \frac{P(B = b \mid A = a_j)P(A = a_j)}{P(B = b)}$$

$$= \frac{P(B = b \mid A = a_j)P(A = a_j)}{\sum_{i=1}^{k} P(B = b \mid A = a_j)P(A = a_j)}$$

Used in practice when we want to compute:

$\mathbf{P}(A \mid B = b)$ for all values of $a_1, a_2, \ldots a_k$

Probabilistic inference

Various inference tasks:

- **Diagnostic task. (from effect to cause)**

$$\mathbf{P}(Pneumonia \mid Fever = T)$$

- **Prediction task. (from cause to effect)**

$$\mathbf{P}(Fever \mid Pneumonia = T)$$

- **Other probabilistic queries** (queries on joint distributions).

$$\mathbf{P}(Fever)$$

$$\mathbf{P}(Fever, ChestPain)$$

Inference

Any query can be computed from the full joint distribution !!!

- **Joint over a subset of variables** is obtained through marginalization

$$P(A=a,C=c)=\sum_i \sum_j P(A=a,B=b_i,C=c,D=d_j)$$

- **Conditional probability over set of variables**, given other variables' values is obtained through marginalization and definition of conditionals

$$P(D=d \mid A=a,C=c)=\frac{P(A=a,C=c,D=d)}{P(A=a,C=c)}$$

$$=\frac{\sum_i P(A=a,B=b_i,C=c,D=d)}{\sum_i \sum_j P(A=a,B=b_i,C=c,D=d_j)}$$

Inference

Any query can be computed from the full joint distribution !!!

- Any joint probability can be expressed as a product of conditionals via the **chain rule**.

$$P(X_1,X_2,\ldots X_n)=P(X_n \mid X_{1,}\ldots X_{n-1})P(X_{1,}\ldots X_{n-1})$$

$$=P(X_n \mid X_{1,}\ldots X_{n-1})P(X_{n-1} \mid X_{1,}\ldots X_{n-2})P(X_{1,}\ldots X_{n-2})$$

$$=\prod_{i=1}^{n} P(X_i \mid X_{1,}\ldots X_{i-1})$$

- Sometimes it is easier to define the distribution in terms of conditional probabilities:
  - E.g. $\mathbf{P}(Fever \mid Pneumonia=T)$
    $\mathbf{P}(Fever \mid Pneumonia=F)$

Modeling uncertainty with probabilities

- Defining the **full joint distribution** makes it possible to represent and reason with uncertainty in a uniform way
- We are able to handle an arbitrary inference problem

Problems:

- **Space complexity.** To store a full joint distribution we need to remember $O(d^n)$ numbers.

  n – number of random variables, d – number of values
- **Inference (time) complexity.** To compute some queries requires $O(d^n)$ steps.
- **Acquisition problem.** Who is going to define all of the probability entries?

Medical diagnosis example

- **Space complexity**.
  - Pneumonia (2 values: T,F), Fever (2: T,F), Cough (2: T,F), WBCcount (3: high, normal, low), paleness (2: T,F)
  - Number of assignments: 2*2*2*3*2=48
  - We need to define at least 47 probabilities.
- **Time complexity**.
  - Assume we need to compute the marginal of Pneumonia=T from the full joint

$$P(Pneumonia = T) =$$
$$= \sum_{i \in T,F} \sum_{j \in T,F} \sum_{k=h,n,l} \sum_{u \in T,F} P(Fever = i, Cough = j, WBCcount = k, Pale = u)$$

  - Sum over: 2*2*3*2=24 combinations

Modeling uncertainty with probabilities

- **Knowledge based system era (70s – early 80's)**
  - **Extensional non-probabilistic models**
  - Solve the space, time and acquisition bottlenecks in probability-based models
  - froze the development and advancement of KB systems and contributed to the slow-down of AI in 80s in general

- Breakthrough (late 80s, beginning of 90s)
  - **Bayesian belief networks**
    - Give solutions to the space, acquisition bottlenecks
    - Partial solutions for time complexities

Bayesian belief networks (BBNs)

Bayesian belief networks.

- Represent the full joint distribution over the variables more compactly with a **smaller number of parameters**.
- Take advantage of **conditional and marginal independences** among random variables

- **A and B are independent**

$$P(A,B) = P(A)P(B)$$

- **A and B are conditionally independent given C**

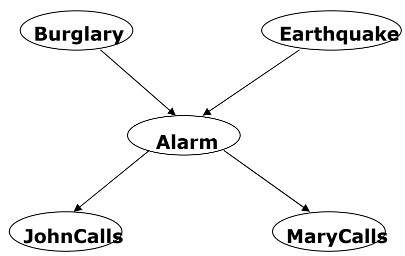
$$P(A,B \mid C) = P(A \mid C)P(B \mid C)$$

$$P(A \mid C,B) = P(A \mid C)$$

Alarm system example.

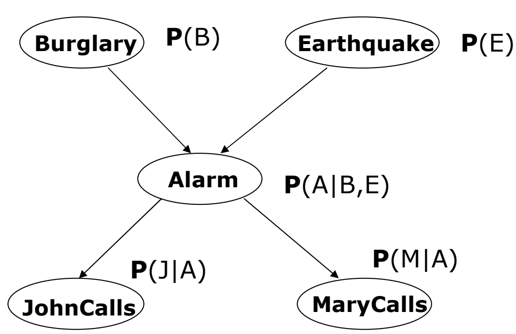
- Assume your house has an **alarm system** against **burglary**. You live in the seismically active area and the alarm system can get occasionally set off by an **earthquake**. You have two neighbors, **Mary** and **John**, who do not know each other. If they hear the alarm they call you, but this is not guaranteed.
- We want to represent the probability distribution of events:
  - Burglary, Earthquake, Alarm, Mary calls and John calls

Causal relations



Bayesian belief network.

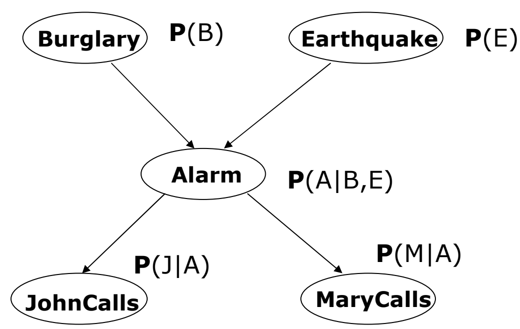
1. **Directed acyclic graph**
   - **Nodes** = random variables
     Burglary, Earthquake, Alarm, Mary calls and John calls
   - **Links** = direct (causal) dependencies between variables.
     The chance of Alarm is influenced by Earthquake, The chance of John calling is affected by the Alarm



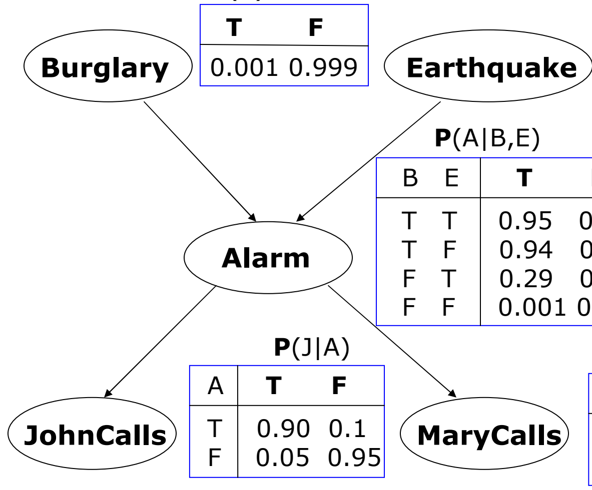
Bayesian belief network.

2. Local conditional distributions

- relate variables and their parents



Bayesian belief network.



P(B)

T	**F**
0.001	0.999

P(E)

T	**F**
0.002	0.998

P(A|B,E)

B	E	**T**	**F**
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

P(J|A)

A	**T**	**F**
T	0.90	0.1
F	0.05	0.95

P(M|A)

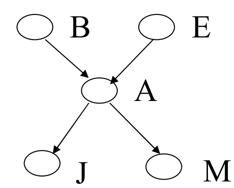
A	T	F
T	0.7	0.3
F	0.01	0.99

Full joint distribution in BBNs

Full joint distribution is defined in terms of local conditional distributions (obtained via the chain rule):

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

Example:



Assume the following assignment of values to random variables

$$B = T, E = T, A = T, J = T, M = F$$

Then its probability is:

$$P(B = T, E = T, A = T, J = T, M = F) =$$

$$P(B = T) P(E = T) P(A = T \mid B = T, E = T) P(J = T \mid A = T) P(M = F \mid A = T)$$