

CS 1571 Introduction to AI

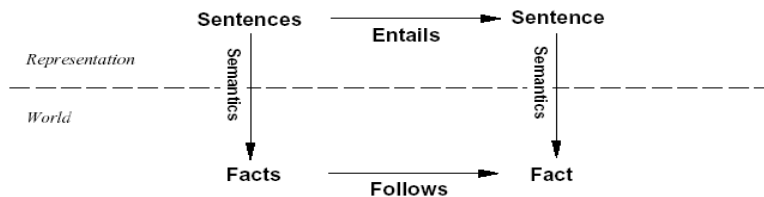
Lecture 12

Propositional logic

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Entailment

- **Entailment** reflects the relation of one fact in the world following from the others



- Entailment $KB \models \alpha$
- Knowledge base KB entails sentence α if and only if α is true in all worlds where KB is true

Sound and complete inference.

Inference is a process by which conclusions are reached.

- We want to implement the inference process on a computer !!

Assume an **inference procedure** i that

- derives a sentence α from the KB : $KB \vdash_i \alpha$

Properties of the inference procedure in terms of entailment

- **Soundness:** An inference procedure is **sound**

If $KB \vdash_i \alpha$ then it is true that $KB \models \alpha$

- **Completeness:** An inference procedure is **complete**

If $KB \models \alpha$ then it is true that $KB \vdash_i \alpha$

Logical inference problem

Logical inference problem:

- **Given:**
 - a knowledge base KB (a set of sentences) and
 - a sentence α (called **a theorem**),
- **Does a KB semantically entail α ?** $KB \models \alpha$?

In other words: In all interpretations in which sentences in the KB are true, is also α true?

Question: Is there a procedure (program) that can decide this problem in a finite number of steps?

Answer: Yes. Logical inference problem for the propositional logic is **decidable**.

Solving logical inference problem

In the following:

How to design the procedure that answers:

$$KB \models \alpha ?$$

Three approaches:

- **Truth-table approach**
- **Inference rules**
- **Conversion to the inverse SAT problem**
 - **Resolution-refutation**

Truth-table approach

Problem: $KB \models \alpha ?$

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False values to propositional symbols)

Example:

		KB		α
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$
True	True	True	True	True
True	False	True	False	False
False	True	True	False	False
False	False	False	True	False

Truth-table approach

Problem: $KB \models \alpha$?

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

Example:

		KB		α
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$
True	True	True	True	True
True	False	True	False	False
False	True	True	False	False
False	False	False	True	False

Truth-table approach

Problem: $KB \models \alpha$?

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

Example:

		KB		α
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$
True	True	True	True	True
True	False	True	False	False
False	True	True	False	False
False	False	False	True	False



Truth-table approach

A two steps procedure:

1. Generate table for all possible interpretations
2. Check whether the sentence α evaluates to true whenever KB evaluates to true

Example: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
<i>True</i>	<i>True</i>	<i>True</i>				
<i>True</i>	<i>True</i>	<i>False</i>				
<i>True</i>	<i>False</i>	<i>True</i>				
<i>True</i>	<i>False</i>	<i>False</i>				
<i>False</i>	<i>True</i>	<i>True</i>				
<i>False</i>	<i>True</i>	<i>False</i>				
<i>False</i>	<i>False</i>	<i>True</i>				
<i>False</i>	<i>False</i>	<i>False</i>				

Truth-table approach

A two steps procedure:

1. Generate table for all possible interpretations
2. Check whether the sentence α evaluates to true whenever KB evaluates to true

Example: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>
<i>True</i>	<i>True</i>	<i>False</i>	<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>
<i>True</i>	<i>False</i>	<i>True</i>	<i>True</i>	<i>False</i>	<i>False</i>	<i>True</i>
<i>True</i>	<i>False</i>	<i>False</i>	<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>
<i>False</i>	<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>
<i>False</i>	<i>True</i>	<i>False</i>	<i>False</i>	<i>True</i>	<i>False</i>	<i>True</i>
<i>False</i>	<i>False</i>	<i>True</i>	<i>True</i>	<i>False</i>	<i>False</i>	<i>False</i>
<i>False</i>	<i>False</i>	<i>False</i>	<i>False</i>	<i>True</i>	<i>False</i>	<i>False</i>

Truth-table approach

A two steps procedure:

1. Generate table for all possible interpretations
2. Check whether the sentence α evaluates to true whenever KB evaluates to true

Example: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	True	True	True	True	True	True
True	True	False	True	True	True	True
True	False	True	True	False	False	True
True	False	False	True	True	True	True
False	True	True	True	True	True	True
False	True	False	False	True	False	True
False	False	True	True	False	False	False
False	False	False	False	True	False	False



Truth-table approach

$$KB = (A \vee C) \wedge (B \vee \neg C) \quad \alpha = (A \vee B)$$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	True	True	True	True	True	True
True	True	False	True	True	True	True
True	False	True	True	False	False	True
True	False	False	True	True	True	True
False	True	True	True	True	True	True
False	True	False	False	True	False	True
False	False	True	True	False	False	False
False	False	False	False	True	False	False

KB entails α

- The **truth-table approach** is **sound and complete** for the propositional logic!!

Limitations of the truth table approach.

$$KB \models \alpha \text{ ?}$$

What is the computational complexity of the truth table approach?

- ?

Limitations of the truth table approach.

$$KB \models \alpha \text{ ?}$$

What is the computational complexity of the truth table approach?

Exponential in the number of the proposition symbols

2^n Rows in the table has to be filled

Limitations of the truth table approach.

$$KB \models \alpha ?$$

What is the computational complexity of the truth table approach?

Exponential in the number of the proposition symbols

2^n Rows in the table has to be filled

But typically only for a small subset of rows the KB is true

Limitations of the truth table approach.

$$KB \models \alpha ?$$

Problem with the truth table approach:

- the truth table is **exponential** in the number of propositional symbols (we checked all assignments)
- KB is true on only a smaller subset

Limitation of the truth table approach.

$$KB \models \alpha ?$$

Problem with the truth table approach:

- the truth table is **exponential** in the number of propositional symbols (we checked all assignments)
- KB is true only on a small subset interpretations

How to make the process more efficient?

Inference rules approach.

$$KB \models \alpha ?$$

Problem with the truth table approach:

- the truth table is **exponential** in the number of propositional symbols (we checked all assignments)
- KB is true on only a smaller subset

How to make the process more efficient?

Solution: check only entries for which KB is *True*.

This is the idea behind the inference rules approach

Inference rules:

- Represent sound inference patterns repeated in inferences
- Can be used to generate new (sound) sentences from the existing ones

Inference rules for logic

- **Modus ponens**

$$\frac{A \Rightarrow B, \quad A}{B}$$

premise

conclusion

- If both sentences in the premise are true then conclusion is true.
- The modus ponens inference rule is **sound**.
 - We can prove this through the truth table.

<i>A</i>	<i>B</i>	<i>A</i> $\Rightarrow$ <i>B</i>
<i>False</i>	<i>False</i>	<i>True</i>
<i>False</i>	<i>True</i>	<i>True</i>
<i>True</i>	<i>False</i>	<i>False</i>
<i>True</i>	<i>True</i>	<i>True</i>

Inference rules for logic

- **And-elimination**

$$\frac{A_1 \wedge A_2 \wedge \dots \wedge A_n}{A_i}$$

- **And-introduction**

$$\frac{A_1, A_2, \dots, A_n}{A_1 \wedge A_2 \wedge \dots \wedge A_n}$$

- **Or-introduction**

$$\frac{A_i}{A_1 \vee A_2 \vee \dots \vee A_i \vee \dots \vee A_n}$$

Inference rules for logic

- Elimination of double negation

$$\frac{\neg\neg A}{A}$$

- Unit resolution

$$\frac{A \vee B, \neg A}{B}$$

- Resolution

$$\frac{A \vee B, \neg B \vee C}{A \vee C}$$

A special
case of



- All of the above inference rules **are sound**. We can prove this through the truth table, similarly to the **modus ponens** case.

Example. Inference rules approach.

KB: $P \wedge Q$ $P \Rightarrow R$ $(Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$

Example. Inference rules approach.

KB: $P \wedge Q$ $P \Rightarrow R$ $(Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P

From 1 and And-elim

$$\frac{A_1 \wedge A_2 \wedge \dots \wedge A_n}{A_i}$$

Example. Inference rules approach.

KB: $P \wedge Q$ $P \Rightarrow R$ $(Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R

From 2,4 and Modus ponens

$$\frac{A \Rightarrow B, \quad A}{B}$$

Example. Inference rules approach.

KB: $P \wedge Q$ $P \Rightarrow R$ $(Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q

From 1 and And-elim

$$\frac{A_1 \wedge A_2 \wedge \dots \wedge A_n}{A_i}$$

Example. Inference rules approach.

KB: $P \wedge Q$ $P \Rightarrow R$ $(Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q
7. $(Q \wedge R)$

From 5,6 and And-introduction

$$\frac{A_1, A_2, \dots, A_n}{A_1 \wedge A_2 \wedge \dots \wedge A_n}$$

Example. Inference rules approach.

KB: $P \wedge Q$ $P \Rightarrow R$ $(Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q
7. $(Q \wedge R)$
8. S

$$\frac{A \Rightarrow B, \quad A}{B}$$

From 7,3 and Modus ponens

Proved: S

Example. Inference rules approach.

KB: $P \wedge Q$ $P \Rightarrow R$ $(Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q
7. $(Q \wedge R)$
8. S

From 1 and And-elim

From 2,4 and Modus ponens

From 1 and And-elim

From 5,6 and And-introduction

From 7,3 and Modus ponens

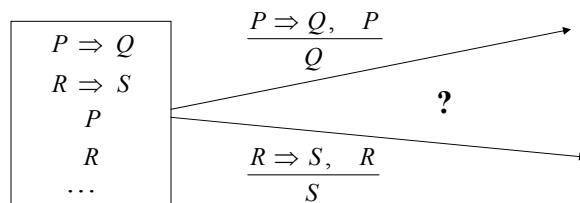
Proved: S

Inference rules

- To show that theorem α holds for a KB
 - we may need to apply a number of sound inference rules

Problem: many possible inference rules to be applied next

Looks familiar?

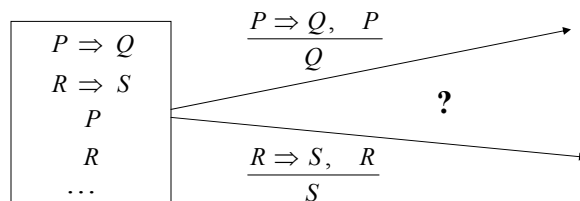


Logic inferences and search

- To show that theorem α holds for a KB
 - we may need to apply a number of sound inference rules

Problem: many possible rules to can be applied next

Looks familiar?



This is an instance of a search problem:

Truth table method (from the search perspective):

- blind enumeration and checking

Logic inferences and search

Inference rule method as a search problem:

- **State:** a set of sentences that are known to be true
- **Initial state:** a set of sentences in the KB
- **Operators:** applications of inference rules
 - Allow us to add new sound sentences to old ones
- **Goal state:** a theorem α is derived from KB

Logic inference:

- **Proof:** A sequence of sentences that are immediate consequences of applied inference rules
- **Theorem proving:** process of finding a proof of theorem

Normal forms

Sentences in the propositional logic can be transformed into one of the normal forms. This can simplify the inferences.

Normal forms used:

Conjunctive normal form (CNF)

- conjunction of clauses (clauses include disjunctions of literals)

$$(A \vee B) \wedge (\neg A \vee \neg C \vee D)$$

Disjunctive normal form (DNF)

- Disjunction of terms (terms include conjunction of literals)

$$(A \wedge \neg B) \vee (\neg A \wedge C) \vee (C \wedge \neg D)$$

Conversion to a CNF

Assume: $\neg(A \Rightarrow B) \vee (C \Rightarrow A)$

1. Eliminate $\Rightarrow, \Leftrightarrow$

$$\neg(\neg A \vee B) \vee (\neg C \vee A)$$

2. Reduce the scope of signs through DeMorgan Laws and double negation

$$(A \wedge \neg B) \vee (\neg C \vee A)$$

3. Convert to CNF using the associative and distributive laws

$$(A \vee \neg C \vee A) \wedge (\neg B \vee \neg C \vee A)$$

and

$$(A \vee \neg C) \wedge (\neg B \vee \neg C \vee A)$$

Satisfiability (SAT) problem

Determine whether a sentence in the conjunctive normal form (CNF) is satisfiable (I.e. can evaluate to true)

$$(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg R \vee S) \wedge (\neg P \vee Q \vee \neg T) \dots$$

It is an instance of a constraint satisfaction problem:

- **Variables:**
 - Propositional symbols (P, R, T, S)
 - Values: *True, False*
- **Constraints:**
 - Every conjunct must evaluate to true, at least one of the literals must evaluate to true
- **A logical inference problem can be solved as a CSP problem. Why?**

Inference problem and satisfiability

Inference problem:

- we want to show that the sentence α is entailed by KB

Satisfiability:

- The sentence is satisfiable if there is some assignment (interpretation) under which the sentence evaluates to true

Connection:

$KB \models \alpha$ if and only if
 $(KB \wedge \neg \alpha)$ is **unsatisfiable**

Consequences:

- inference problem is NP-complete
- programs for solving the SAT problem can be used to solve the inference problem