CS 1571 Introduction to AI
Lecture 23

Bayesian belief networks
Inference

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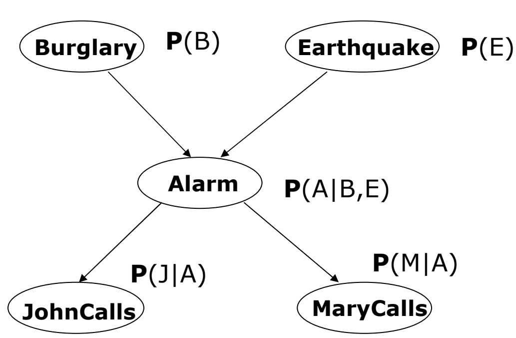
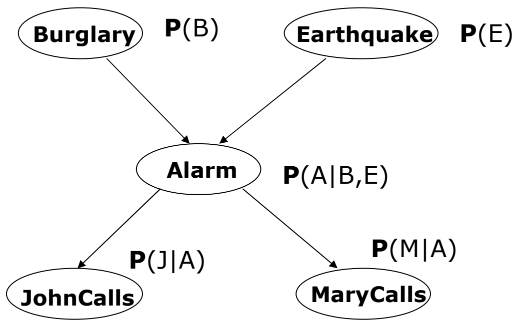
Bayesian belief network.

1. **Directed acyclic graph**
   - **Nodes** = random variables
     Burglary, Earthquake, Alarm, Mary calls and John calls
   - **Links** = direct (causal) dependencies between variables.
     The chance of Alarm is influenced by Earthquake, The chance of John calling is affected by the Alarm



Bayesian belief network.

2. **Local conditional distributions**
   - relate variables and their parents



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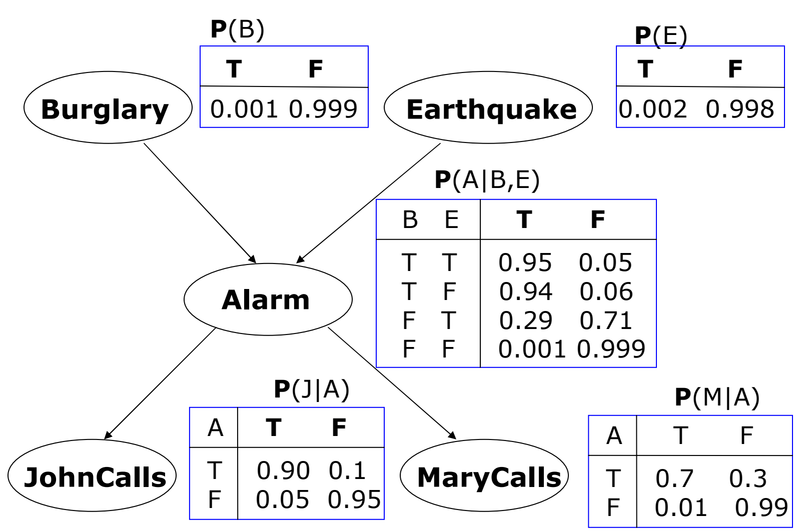


P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

P(J|A)

A	T	F
T	0.90	0.1
F	0.05	0.95

P(M|A)

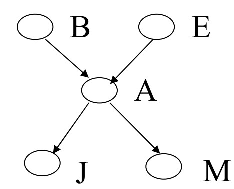
A	T	F
T	0.7	0.3
F	0.01	0.99

Full joint distribution in BBNs

Full joint distribution is defined in terms of local conditional distributions (obtained via the chain rule):

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

Example:



Assume the following assignment of values to random variables

$B = T, E = T, A = T, J = T, M = F$

Then its probability is:

$$P(B = T, E = T, A = T, J = T, M = F) =$$
$$P(B = T)P(E = T)P(A = T \mid B = T, E = T)P(J = T \mid A = T)P(M = F \mid A = T)$$

Parameter complexity problem

- In the BBN the **full joint distribution** is defined as:

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

- **What did we save?**

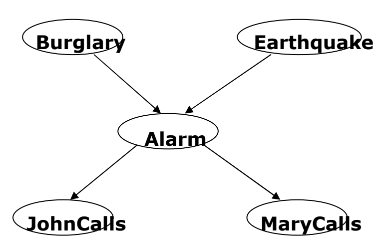
Alarm example: 5 binary (True, False) variables

# of parameters of the full joint:

$$2^5 = 32$$

One parameter is for free:

$$2^5 - 1 = 31$$



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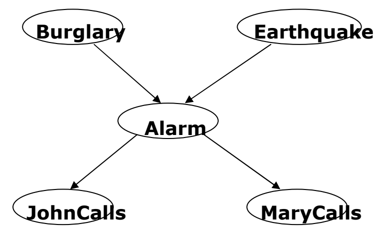
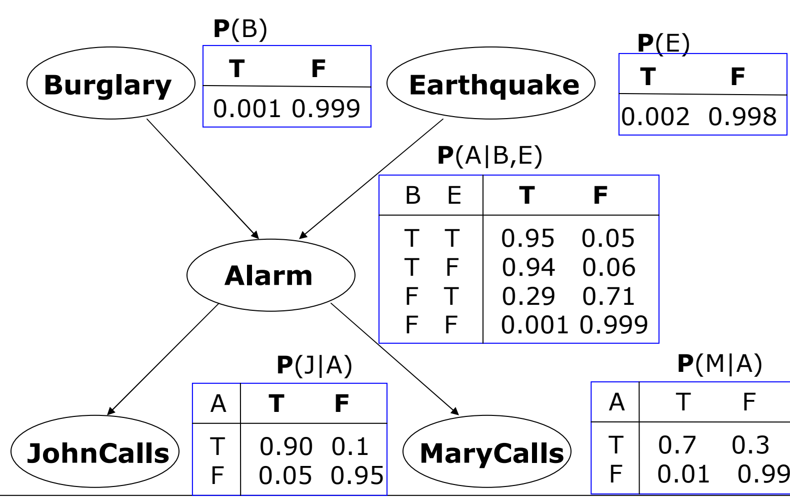


Bayesian belief network.

- In the BBN the **full joint distribution** is expressed using a set of local conditional distributions



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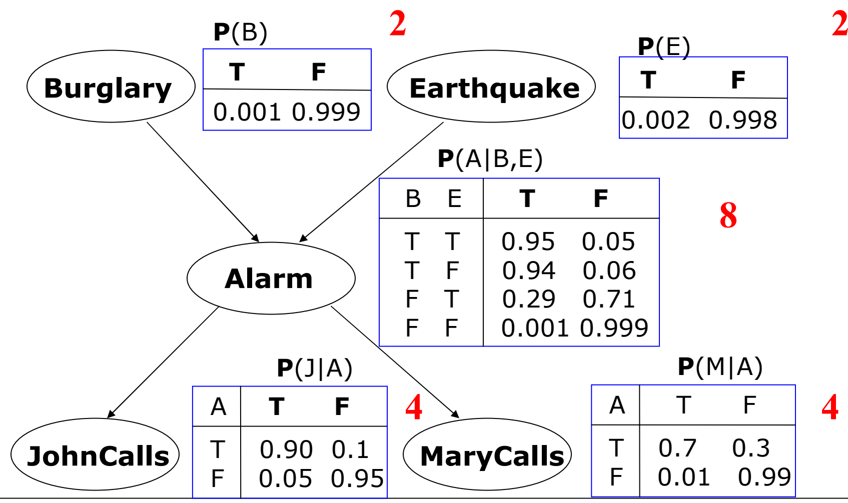

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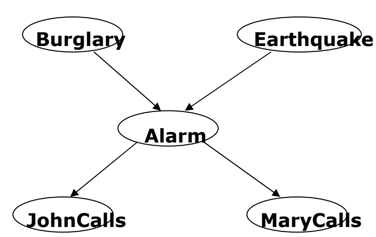
# of parameters of the BBN:

$$2^3 + 2(2^2) + 2(2) = 20$$

One parameter in every conditional is for free:

?



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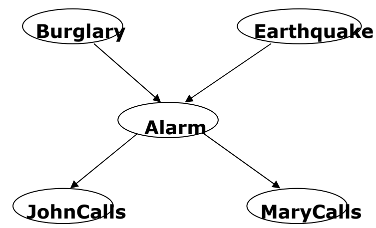


# of parameters of the BBN:

$$2^3 + 2(2^2) + 2(2) = 20$$

One parameter in every conditional is for free:

$$2^2 + 2(2) + 2(1) = 10$$

Inference in Bayesian networks

- BBN models compactly the full joint distribution by taking advantage of existing independences between variables
- Simplifies the acquisition of a probabilistic model
- But we are interested in solving various **inference tasks**:
  - **Diagnostic task. (from effect to cause)**

    $$\mathbf{P}(Burglary \mid JohnCalls = T)$$

  - **Prediction task. (from cause to effect)**

    $$\mathbf{P}(JohnCalls \mid Burglary = T)$$

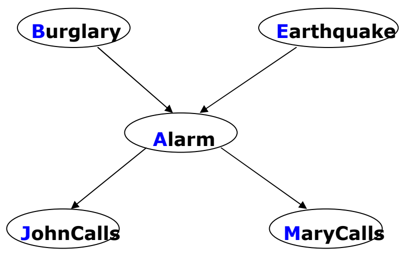
  - **Other probabilistic queries** (queries on joint distributions).

    $$\mathbf{P}(Alarm)$$

- **Main issue:** Can we take advantage of independences to construct special algorithms and speeding up the inference?

Inference in Bayesian network

- **Bad news:**
  - Exact inference problem in BBNs is NP-hard (Cooper)
  - Approximate inference is NP-hard (Dagum, Luby)
- **But** very often we can achieve significant improvements
- Assume our Alarm network



- Assume we want to compute: $P(J = T)$

Inference in Bayesian networks

Computing: $P(J = T)$

Approach 1. Blind approach.

- Sum out all un-instantiated variables from the full joint,
- express the joint distribution as a product of conditionals

$$P(J = T) =$$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(B = b, E = e, A = a, J = T, M = m)$$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(J = T \mid A = a) P(M = m \mid A = a) P(A = a \mid B = b, E = e) P(B = b) P(E = e)$$

Computational cost:

Number of additions: ?

Number of products: ?

Inference in Bayesian networks

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Computational cost:

Number of additions: 15

Number of products: ?

Inference in Bayesian networks

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Computational cost:

Number of additions: 15

Number of products: 16*4=64

Inference in Bayesian networks

Approach 2. Interleave sums and products

- Combines sums and product in a smart way (multiplications by constants can be taken out of the sum)

$$P(J=T)=$$

$$=\sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F}P(J=T\,|\,A=a)P(M=m\,|\,A=a)P(A=a\,|B=b,E=e)P(B=b)P(E=e)$$

$$=\sum_{b\in T,F}\sum_{a\in T.F}\sum_{m\in T,F}P(J=T\,|\,A=a)P(M=m\,|\,A=a)P(B=b)[\sum_{e\in T,F}P(A=a\,|\,B=b,E=e)P(E=e)]$$

$$=\sum_{a\in T,F}P(J=T\,|\,A=a)[\sum_{m\in T,F}P(M=m\,|\,A=a)][\sum_{b\in T,F}P(B=b)[\sum_{e\in T,F}P(A=a\,|\,B=b,E=e)P(E=e)]]$$

Computational cost:

Number of additions: 1+2*[1+1+2*1]=**?**

Number of products: 2*[2+2*(1+2*1)]=?

Inference in Bayesian networks

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$$=\sum_{a\in T,F}P(J=T\,|\,A=a)[\sum_{m\in T,F}P(M=m\,|\,A=a)][\sum_{b\in T,F}P(B=b)[\sum_{e\in T,F}P(A=a\,|\,B=b,E=e)P(E=e)]]$$

Computational cost:

Number of additions: 1+2*[1+1+2*1]=**9**

Number of products: 2*[2+2*(1+2*1)]=?

Inference in Bayesian networks

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Computational cost:

Number of additions: 1+2*[1+1+2*1]=**9**

Number of products: 2*[2+2*(1+2*1)]=16

Inference in Bayesian networks

- The smart interleaving of sums and products can help us to speed up the computation of joint probability queries
- What if we want to compute: $P(B=T,J=T)$

$$P(B=T,J=T)=$$

$$=\sum_{a\in T,F}P(J=T\mid A=a)[\sum_{m\in T,F}P(M=m\mid A=a)]\left[P(B=T)\left[\sum_{e\in T,F}P(A=a\mid B=T,E=e)P(E=e)\right]\right]$$

$$P(J=T)=$$

$$=\sum_{a\in T,F}P(J=T\mid A=a)[\sum_{m\in T,F}P(M=m\mid A=a)]\left[\sum_{b\in T,F}P(B=b)\left[\sum_{e\in T,F}P(A=a\mid B=b,E=e)P(E=e)\right]\right]$$

- A lot of shared computation
  - Smart cashing of results can save the time for more queries

Inference in Bayesian networks

- When cashing of results becomes handy?
- What if we want to compute a diagnostic query:

$$P(B=T \mid J=T) = \frac{P(B=T, J=T)}{P(J=T)}$$

- Exactly probabilities we have just compared !!
- There are other queries when cashing and ordering of sums and products can be shared and saves computation

$$\mathbf{P}(B \mid J=T) = \frac{\mathbf{P}(B, J=T)}{P(J=T)} = \alpha \mathbf{P}(B, J=T)$$

- General technique: **Recursive decomposition**

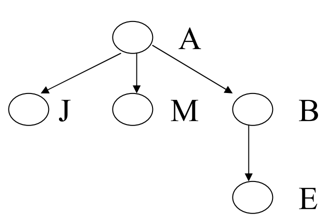
Inference in Bayesian networks

General idea:

$$P(True) = 1 =$$

$$= \sum_{a \in T,F} \underbrace{[\sum_{j \in T,F} P(J=j \mid A=a)]}_{f_J(a)} \underbrace{[\sum_{m \in T,F} P(M=m \mid A=a)]}_{f_M(a)} \underbrace{[\sum_{b \in T,F} P(B=b) \underbrace{[\sum_{e \in T,F} P(A=a \mid B=b, E=e) P(E=e)]}_{f_E(a,b)}}_{f_B(a)}$$

Recursive decomposition:



Results cashed in the tree structure

Variable elimination

- **Recursive decomposition:**
  - Interleave sum and products before inference
- **Variable elimination:**
  - Similar idea but interleave sum and products one variable at the time during inference
  - E.g. Query $P(J=T)$ requires to eliminate A,B,E,M and this can be done in different order

$$P(J=T)=$$

$$=\sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F}P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

Variable elimination

Assume order: M, E, B,A to calculate $P(J=T)$

$$=\sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F}P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

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$$=\sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}P(J=T\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)\quad 1$$

$$=\sum_{a\in T,F}\sum_{b\in T,F}P(J=T\mid A=a)P(B=b)\left[\sum_{e\in T,F}P(A=a\mid B=b,E=e)P(E=e)\right]$$

$$=\sum_{a\in T,F}\sum_{b\in T,F}P(J=T\mid A=a)P(B=b)\,\tau_1(A=a,B=b)$$

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$$=\sum_{a\in T,F}P(J=T\mid A=a)\quad\tau_2(A=a)$$

Inference in Bayesian network

- **Exact inference algorithms:**
  - **Variable elimination** (Book)
  - **Recursive decomposition** (Cooper, Darwiche)
  - Symbolic inference (D'Ambrosio)
  - Belief propagation algorithm (Pearl)
  - **Clustering and joint tree approach (Lauritzen, Spiegelhalter)** (Book)
  - Arc reversal (Olmsted, Schachter)

- **Approximate inference algorithms:**
  - **Monte Carlo methods:** (Book)
    - Forward sampling, Likelihood sampling
  - Variational methods

Monte Carlo approaches

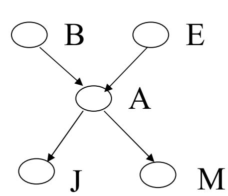
- **MC approximation**:
  - The probability is approximated using sample frequencies
  - **Example:**

$$\tilde{P}(B=T, J=T) = \frac{N_{B=T,J=T}}{N}$$

$N_{B=T,J=T}$ ← $\# \, samples \;\; with \;\; B=T, J=T$

N ← $total \;\; \# \; samples$

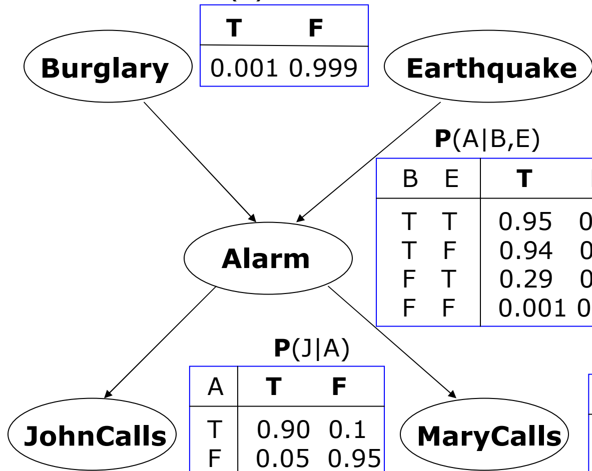
- **BBN sampling:**



Generate sample in a top down manner, following the links

- **One sample gives one assignment of values to all variables**

BBN sampling example



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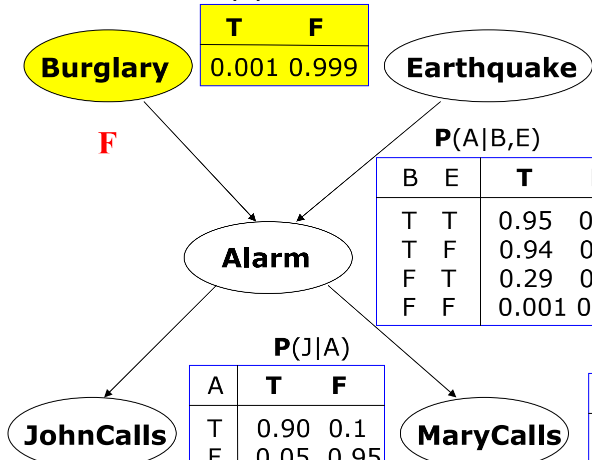

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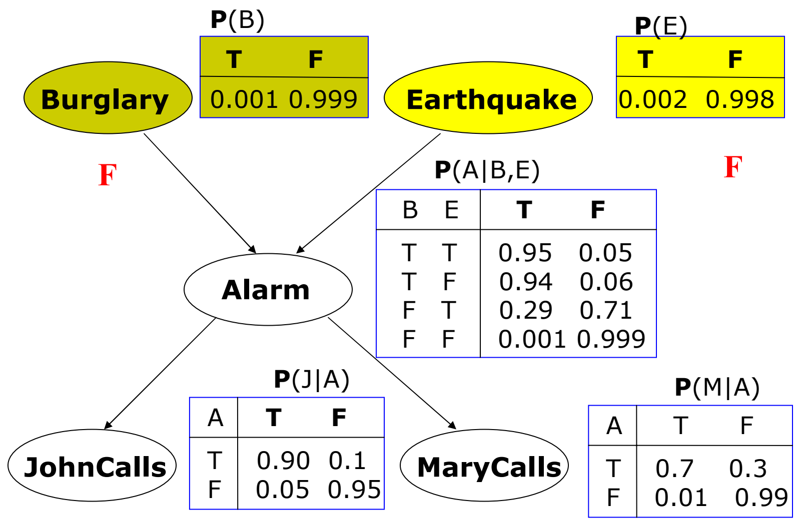

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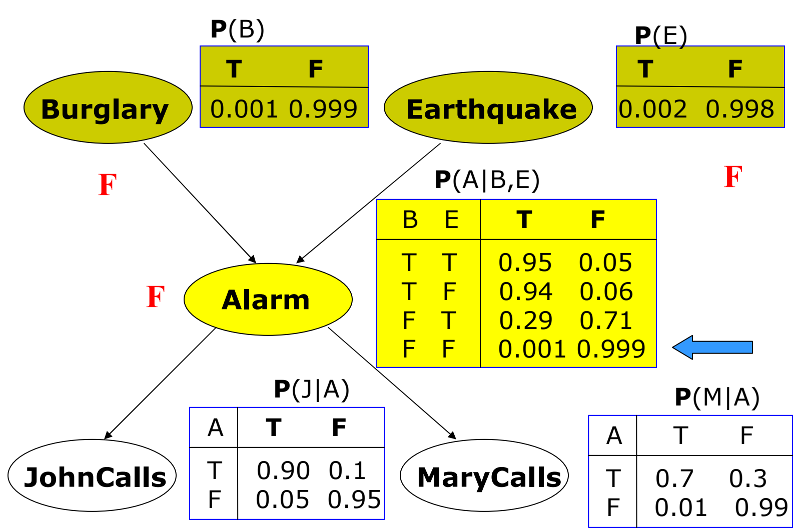

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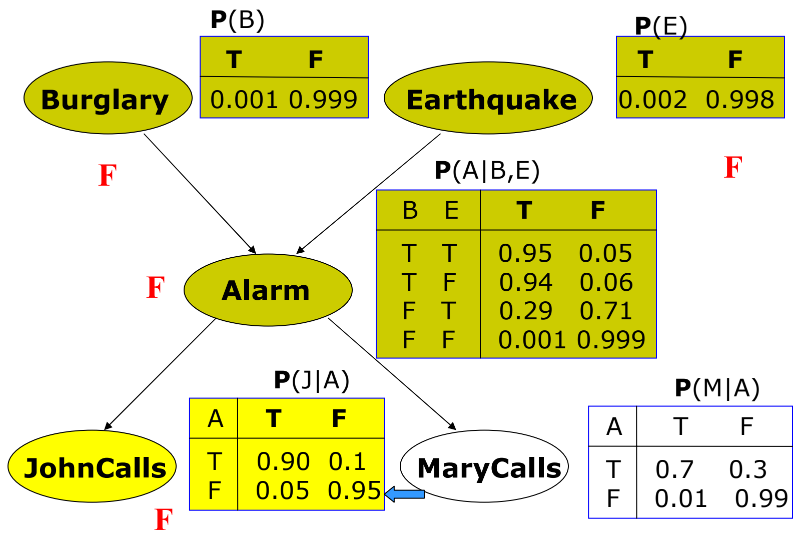

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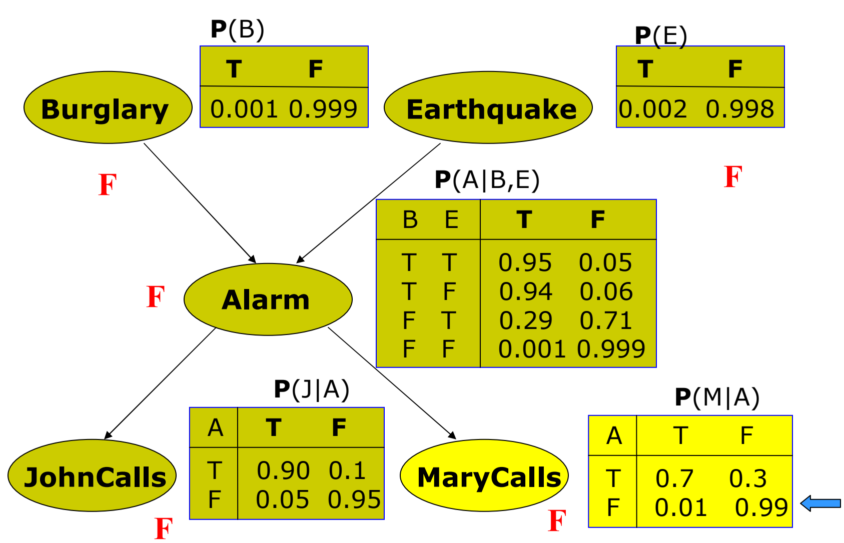
P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

BBN sampling example



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

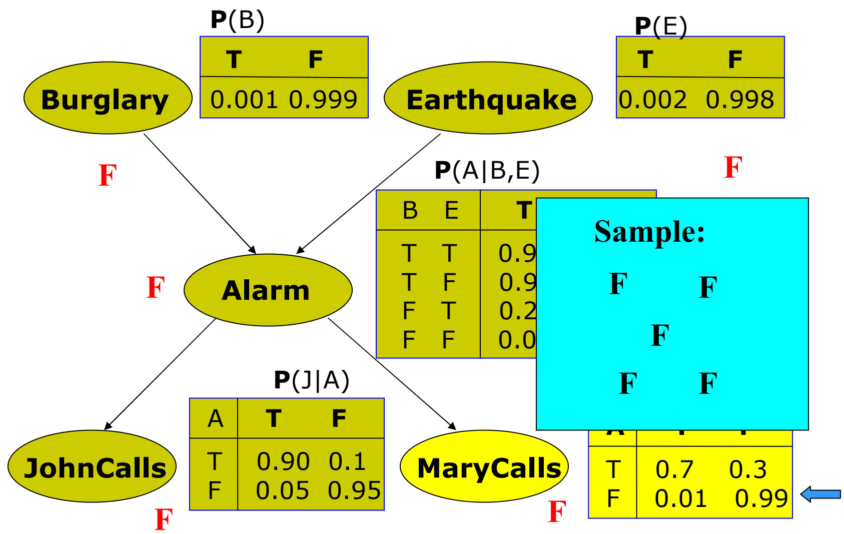
P(J|A)

A	T	F
T	0.90	0.1
F	0.05	0.95

P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

BBN sampling example



Monte Carlo approaches

- **MC approximation of conditional probabilities**:
  - The probability is approximated using sample frequencies
  - **Example:**

$$\tilde{P}(B=T \mid J=T) = \frac{N_{B=T,J=T}}{N_{J=T}}$$

$N_{B=T,J=T}$: $\#\, samples\ \ with\ \ B=T, J=T$

$N_{J=T}$: $\#\, samples\ \ with\ \ J=T$

- **Rejection sampling:**
  - Generate samples from the full joint by sampling BBN
  - Use only samples that agree with the condition, the remaining samples are rejected
- **Problem:** many samples can be rejected