

CS 1571 Introduction to AI

Lecture 15

First-order logic

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Limitations of propositional logic

World we want to represent and reason about consists of a number of objects with variety of properties and relations among them

Propositional logic:

- Represents statements about the world without reflecting this structure and without modeling these entities explicitly

Consequence:

- some knowledge is hard or impossible to encode in the propositional logic.
- Two cases that are hard to represent:
 - **Statements about similar objects, relations**
 - **Statements referring to groups of objects.**

Limitations of propositional logic

- **Statements about similar objects and relations needs to be enumerated**

- **Example:** Seniority of people domain

For inferences we need:

John is older than Mary $\wedge$ *Mary is older than Paul*

$\Rightarrow$ *John is older than Paul*

Jane is older than Mary $\wedge$ *Mary is older than Paul*

$\Rightarrow$ *Jane is older than Paul*

- **Problem:** if we have many people and facts about their seniority we need represent many rules like this to allow inferences

- **Possible solution:** **introduce variables**

PersA is older than *PersB* $\wedge$ *PersB* is older than *PersC*

$\Rightarrow$ *PersA* is older than *PersC*

Limitations of propositional logic

- **Statements referring to groups of objects require exhaustive enumeration of objects**

- **Example:**

Assume we want to express *Every student likes vacation*

Doing this in propositional logic would require to include statements about every student

John likes vacation $\wedge$

Mary likes vacation $\wedge$

Ann likes vacation $\wedge$

...

- **Solution:** Allow quantification in statements

First-order logic (FOL)

- More expressive than **propositional logic**
- **Eliminates deficiencies of PL by:**
 - Representing objects, their properties, relations and statements about them;
 - Introducing variables that refer to an arbitrary objects and can be substituted by a specific object
 - Introducing quantifiers allowing us to make statements over groups objects without the need to represent each of them separately

Logic

Logic is defined by:

- **A set of sentences**
 - A sentence is constructed from a set of primitives according to syntax rules.
- **A set of interpretations**
 - An interpretation gives a semantic to primitives. It associates primitives with objects, values in the real world.
- **The valuation (meaning) function V**
 - Assigns a truth value to a given sentence under some interpretation

$$V : \text{sentence} \times \text{interpretation} \rightarrow \{True, False\}$$

First-order logic. Syntax.

Term - syntactic entity for representing objects

Terms in FOL:

- **Constant symbols:** represent specific objects
 - E.g. *John*, *France*, *car89*
- **Variables:** represent objects of a certain type (type = domain of discourse)
 - E.g. *x*, *y*, *z*
- **Functions** applied to one or more terms
 - E.g. *father-of* (*John*)
father-of(*father-of*(*John*))

First order logic. Syntax.

Sentences in FOL:

- **Atomic sentences:**
 - A **predicate symbol** applied to 0 or more terms

Examples:

Red(*car12*),
Sister(*Amy*, *Jane*);
Manager(*father-of*(*John*));

- $t_1 = t_2$ **equivalence** of terms

Example:

John = *father-of*(*Peter*)

First order logic. Syntax.

Sentences in FOL:

- **Complex sentences:**

- Assume ϕ, ψ are sentences in FOL. Then:

$$- (\phi \wedge \psi) \quad (\phi \vee \psi) \quad (\phi \Rightarrow \psi) \quad (\phi \Leftrightarrow \psi) \quad \neg \psi$$

and

$$- \quad \forall x \phi \quad \exists y \phi$$

are sentences

Symbols $\exists, \forall$

- stand for the **existential** and the **universal** quantifier

Semantics. Interpretation.

An interpretation I is defined by a **mapping** to the **domain of discourse D** or **relations on D**

- **domain of discourse:** a set of objects in the world we represent and refer to;

An interpretation I maps:

- Constant symbols to objects in D

$$I(\text{John}) = \text{stick figure}$$

- Predicate symbols to relations, properties on D

$$I(\text{brother}) = \{ \langle \text{stick figure}, \text{stick figure with glasses} \rangle; \langle \text{stick figure}, \text{stick figure with glasses} \rangle; \dots \}$$

- Function symbols to functional relations on D

$$I(\text{father-of}) = \{ \langle \text{stick figure} \rangle \rightarrow \text{stick figure}; \langle \text{stick figure} \rangle \rightarrow \text{stick figure with glasses}; \dots \}$$

Semantics of sentences.

Meaning (evaluation) function:

$$V : \text{sentence} \times \text{interpretation} \rightarrow \{True, False\}$$

A **predicate** $predicate(term-1, term-2, term-3, term-n)$ is true for the interpretation I , iff the objects referred to by $term-1$, $term-2$, $term-3$, $term-n$ are in the relation referred to by $predicate$

$$I(John) = \text{stick figure} \quad I(Paul) = \text{robot stick figure}$$

$$I(brother) = \{ \langle \text{stick figure}, \text{robot stick figure} \rangle; \langle \text{robot stick figure}, \text{stick figure} \rangle; \dots \}$$

$$brother(John, Paul) = \langle \text{stick figure}, \text{robot stick figure} \rangle \text{ in } I(brother)$$

$$V(brother(John, Paul), I) = True$$

Semantics of sentences.

- **Equality** $V(term-1 = term-2, I) = True$

Iff $I(term-1) = I(term-2)$

- **Boolean expressions: standard**

E.g. $V(sentence-1 \vee sentence-2, I) = True$

Iff $V(sentence-1, I) = True$ or $V(sentence-2, I) = True$

- **Quantifications**

$$V(\forall x \phi, I) = True \quad \text{substitution of } x \text{ with } d$$

Iff for all $d \in D$ $V(\phi, I[x/d]) = True$

$$V(\exists x \phi, I) = True$$

Iff there is a $d \in D$, s.t. $V(\phi, I[x/d]) = True$

Sentences with quantifiers

- **Universal quantification**

All Upitt students are smart

- **Assume the universe of discourse of x are Upitt students**

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$\forall x \text{ smart}(x)$

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- **Assume the universe of discourse of x are people**

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- **Assume the universe of discourse of x are people**

$$\forall x \text{ student}(x) \wedge \text{at}(x, \text{Upitt}) \Rightarrow \text{smart}(x)$$

Sentences with quantifiers

- **Universal quantification**

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$$\forall x \text{ smart}(x)$$

- Assume the universe of discourse of x are students

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Typically the universal quantifier connects with an implication

Sentences with quantifiers

- **Existential quantification**

Someone at CMU is smart

- Assume the universe of discourse of x are CMU affiliates

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$$\exists x \text{ smart}(x)$$

- **Assume the universe of discourse of x are people**

$$\exists x \text{ at}(x, \text{CMU}) \wedge \text{smart}(x)$$

Sentences with quantifiers

- **Existential quantification**

Someone at CMU is smart

- **Assume the universe of discourse of x are CMU affiliates**

$$\exists x \text{ smart}(x)$$

- **Assume the universe of discourse of x are people**

$$\exists x \text{ at}(x, \text{CMU}) \wedge \text{smart}(x)$$

Typically the existential quantifier connects with a conjunction

Translation with quantifiers

- Assume two predicates $S(x)$ and $P(x)$

Universal statements typically tie with implications

- All $S(x)$ is $P(x)$
 - $\forall x (S(x) \rightarrow P(x))$
- No $S(x)$ is $P(x)$
 - $\forall x (S(x) \rightarrow \neg P(x))$

Existential statements typically tie with conjunction

- Some $S(x)$ is $P(x)$
 - $\exists x (S(x) \wedge P(x))$
- Some $S(x)$ is not $P(x)$
 - $\exists x (S(x) \wedge \neg P(x))$

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- There is a person who loves everybody.
- **Translation:**
 - Assume:
 - Variables x and y denote people
 - A predicate $L(x,y)$ denotes: “ x loves y ”
- Then we can write in the predicate logic:
 - ?

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- There is a person who loves everybody.
- **Translation:**
 - Assume:
 - Variables x and y denote people
 - A predicate L(x,y) denotes: “x loves y”
- Then we can write in the predicate logic:

$$\exists x \forall y \text{ L(x,y)}$$

Translation exercise

Suppose:

- Variables x, y denote people
- $L(x, y)$ denotes “ x loves y ”.

Translate:

- Everybody loves Raymond. ?

Translation exercise

Suppose:

- Variables x, y denote people
- $L(x, y)$ denotes “ x loves y ”.

Translate:

- Everybody loves Raymond. $\forall x L(x, \text{Raymond})$
- Everybody loves somebody. $?$

Translation exercise

Suppose:

- Variables x, y denote people
- $L(x, y)$ denotes “ x loves y ”.

Translate:

- Everybody loves Raymond. $\forall x L(x, \text{Raymond})$
- Everybody loves somebody. $\forall x \exists y L(x, y)$
- There is somebody whom everybody loves. $?$

Translation exercise

Suppose:

- Variables x, y denote people
- $L(x, y)$ denotes “ x loves y ”.

Translate:

- Everybody loves Raymond. $\forall x L(x, \text{Raymond})$
- Everybody loves somebody. $\forall x \exists y L(x, y)$
- There is somebody whom everybody loves. $\exists y \forall x L(x, y)$
- There is somebody who Raymond doesn't love. ?

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Translate:

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- Everybody loves somebody. $\forall x \exists y L(x, y)$
- There is somebody whom everybody loves. $\exists y \forall x L(x, y)$
- There is somebody who Raymond doesn't love.
 $\exists y \neg L(\text{Raymond}, y)$
- There is somebody whom no one loves. ?

Translation exercise

Suppose:

- Variables x, y denote people
- $L(x, y)$ denotes “ x loves y ”.

Translate:

- Everybody loves Raymond. $\forall x L(x, \text{Raymond})$
- Everybody loves somebody. $\forall x \exists y L(x, y)$
- There is somebody whom everybody loves. $\exists y \forall x L(x, y)$
- There is somebody who Raymond doesn't love.
 $\exists y \neg L(\text{Raymond}, y)$
- There is somebody whom no one loves.
 $\exists y \forall x \neg L(x, y)$

Order of quantifiers

- **Order of quantifiers of the same type does not matter**

For all x and y , if x is a parent of y then y is a child of x

$$\forall x, y \text{ parent } (x, y) \Rightarrow \text{child } (y, x)$$

$$\forall y, x \text{ parent } (x, y) \Rightarrow \text{child } (y, x)$$

- **Order of different quantifiers changes the meaning**

$$\forall x \exists y \text{ loves } (x, y)$$

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- **Order of different quantifiers changes the meaning**

$$\forall x \exists y \text{ loves } (x, y)$$

Everybody loves somebody

$$\exists y \forall x \text{ loves } (x, y)$$

Order of quantifiers

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For all x and y , if x is a parent of y then y is a child of x

$$\forall x, y \text{ parent } (x, y) \Rightarrow \text{child } (y, x)$$

$$\forall y, x \text{ parent } (x, y) \Rightarrow \text{child } (y, x)$$

- **Order of different quantifiers changes the meaning**

$$\forall x \exists y \text{ loves } (x, y)$$

Everybody loves somebody

$$\exists y \forall x \text{ loves } (x, y)$$

There is someone who is loved by everyone

Connections between quantifiers

Everyone likes ice cream

?

Connections between quantifiers

Everyone likes ice cream

$\forall x \text{ likes } (x, \text{IceCream})$

Connections between quantifiers

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Is it possible to convey the same meaning using an existential quantifier ?

Connections between quantifiers

Everyone likes ice cream

$\forall x \text{ likes } (x, \text{IceCream})$

Is it possible to convey the same meaning using an existential quantifier ?

There is no one who does not like ice cream

$\neg \exists x \neg \text{likes } (x, \text{IceCream})$

A universal quantifier in the sentence can be expressed using an existential quantifier !!!

Connections between quantifiers

Someone likes ice cream

?

Connections between quantifiers

Someone likes ice cream

$\exists x \text{ likes } (x, \text{IceCream})$

Is it possible to convey the same meaning using a universal quantifier ?

Connections between quantifiers

Someone likes ice cream

$\exists x \text{ likes } (x, \text{IceCream})$

Is it possible to convey the same meaning using a universal quantifier ?

Not everyone does not like ice cream

$\neg \forall x \neg \text{likes } (x, \text{IceCream})$

An existential quantifier in the sentence can be expressed using a universal quantifier !!!

Representing knowledge in FOL

Example:

Kinship domain

- **Objects:** people

John, Mary, Jane, ...

- **Properties:** gender

Male(x), Female(x)

- **Relations:** parenthood, brotherhood, marriage

Parent(x, y), Brother(x, y), Spouse(x, y)

- **Functions:** mother-of (one for each person x)

MotherOf(x)

Kinship domain in FOL

Relations between predicates and functions: write down what we know about them; how relate to each other.

- Male and female are disjoint categories

$$\forall x \text{ Male}(x) \Leftrightarrow \neg \text{Female}(x)$$

- Parent and child relations are inverse

$$\forall x, y \text{ Parent}(x, y) \Leftrightarrow \text{Child}(y, x)$$

- A grandparent is a parent of parent

$$\forall g, c \text{ Grandparent}(g, c) \Leftrightarrow \exists p \text{ Parent}(g, p) \wedge \text{Parent}(p, c)$$

- A sibling is another child of one's parents

$$\forall x, y \text{ Sibling}(x, y) \Leftrightarrow (x \neq y) \wedge \exists p \text{ Parent}(p, x) \wedge \text{Parent}(p, y)$$

- And so on