CS 1571 Introduction to AI
Lecture 13

Inference in propositional logic

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Logical inference problem

Logical inference problem:

- **Given:**
  - a knowledge base KB (a set of sentences) and
  - a sentence α (called **a theorem**),
- **Does a KB semantically entail** α ? $KB \models \alpha$?

In other words: In all interpretations in which sentences in the KB are true, is also α true?

Question: Is there a procedure (program) that can decide this problem in a finite number of steps?

Answer: **Yes**. Logical inference problem for the propositional logic is **decidable**.

Solving logical inference problem

In the following:

How to design the procedure that answers:

$$KB \models \alpha \ ?$$

Three approaches:

- **Truth-table approach**
- **Inference rules**
- **Conversion to the inverse SAT problem**
  - **Resolution-refutation**

Truth-table approach

A two steps procedure:

1. **Generate table for all possible interpretations**
2. Check whether the sentence α evaluates to true whenever KB evaluates to true

Example: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\quad \alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	*True*	*True*				
True	*True*	*False*				
True	*False*	*True*				
True	*False*	*False*				
False	*True*	*True*				
False	*True*	*False*				
False	*False*	*True*				
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True	*False*	*True*	*True*	*False*	*False*	*True*
True	*False*	*False*	*True*	*True*	*True*	*True*
False	*True*	*True*	*True*	*True*	*True*	*True*
False	*True*	*False*	*False*	*True*	*False*	*True*
False	*False*	*True*	*True*	*False*	*False*	*False*
False	*False*	*False*	*False*	*True*	*False*	*False*

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True	*True*	*False*	*True*	*True*	*True*	*True*	✓
True	*False*	*True*	*True*	*False*	*False*	*True*	
True	*False*	*False*	*True*	*True*	*True*	*True*	✓
False	*True*	*True*	*True*	*True*	*True*	*True*	✓
False	*True*	*False*	*False*	*True*	*False*	*True*	
False	*False*	*True*	*True*	*False*	*False*	*False*	
False	*False*	*False*	*False*	*True*	*False*	*False*	

Truth-table approach

$$KB = (A \vee C) \wedge (B \vee \neg C) \qquad \alpha = (A \vee B)$$

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True	*True*	*True*	*True*	*True*	*True*	*True*
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True	*False*	*True*	*True*	*False*	*False*	*True*
True	*False*	*False*	*True*	*True*	*True*	*True*
False	*True*	*True*	*True*	*True*	*True*	*True*
False	*True*	*False*	*False*	*True*	*False*	*True*
False	*False*	*True*	*True*	*False*	*False*	*False*
False	*False*	*False*	*False*	*True*	*False*	*False*

KB entails α

- The **truth-table approach** is **sound and complete** for the propositional logic!!

Limitations of the truth table approach.

$$KB \models \alpha \ ?$$

What is the computational complexity of the truth table approach?

- ?

Limitations of the truth table approach.

$KB \models \alpha$?

What is the computational complexity of the truth table approach?

Exponential in the number of the proposition symbols

2^n Rows in the table has to be filled

But typically only for a small subset of rows the KB is true

Inference rules approach.

$KB \models \alpha$?

Problem with the truth table approach:

- the truth table is **exponential** in the number of propositional symbols (we checked all assignments)
- KB is true on only a smaller subset

How to make the process more efficient?

Solution: check only entries for which KB is *True*.

This is the idea behind **the inference rules approach**

Inference rules:

- Represent sound inference patterns repeated in inferences
- Can be used to generate new (sound) sentences from the existing ones

Inference rules for logic

- **Modus ponens**

$$\frac{A \Rightarrow B, \quad A}{B}$$ ← **premise** / ← **conclusion**

- If both sentences in the premise are true then conclusion is true.
- The modus ponens inference rule is **sound.**
  - We can prove this through the truth table.

A	B	$A \Rightarrow B$
False	*False*	*True*
False	*True*	*True*
True	*False*	*False*
True	*True*	*True*

Inference rules for logic

- **Elimination of double negation** $$\frac{\neg\neg A}{A}$$

- **Unit resolution** $$\frac{A \vee B, \quad \neg A}{B}$$

A special case of

- **Resolution** $$\frac{A \vee B, \quad \neg B \vee C}{A \vee C}$$

- All of the above inference rules **are sound.** We can prove this through the truth table, similarly to the **modus ponens** case.

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$

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3. $(Q \wedge R) \Rightarrow S$
4. P **From 1 and And-elim**

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

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1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P — **From 1 and And-elim**
5. R — **From 2,4 and Modus ponens**
6. Q — **From 1 and And-elim**
7. $(Q \wedge R)$ — **From 5,6 and And-introduction**
8. S — **From 7,3 and Modus ponens**

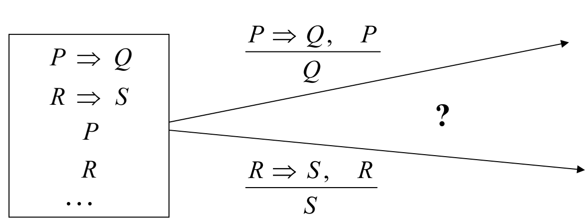
Proved: S

Inference rules

- To show that theorem α holds for a KB
  - we may need to apply a number of sound inference rules

Problem: many possible inference rules to be applied next

Looks familiar?



Logic inferences and search

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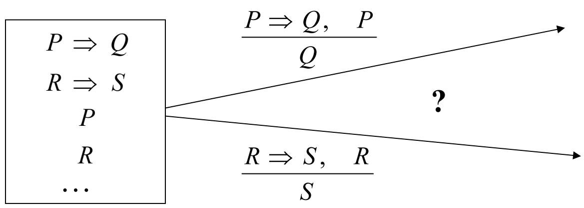

This is an instance of a search problem:

Truth table method (from the search perspective):

- blind enumeration and checking

Logic inferences and search

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  - we may need to apply a number of sound inference rules

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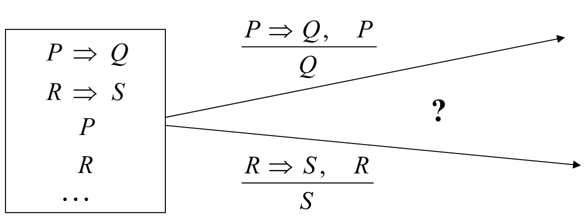

This is an instance of a search problem:

Truth table method (from the search perspective):

- blind enumeration and checking

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Logic inferences and search

Inference rule method as a search problem:

- **State:** a set of sentences that are known to be true
- **Initial state**: a set of sentences in the KB
- **Operators**: applications of inference rules
  - Allow us to add new sound sentences to old ones
- **Goal state:** a theorem α is derived from KB

Logic inference:

- **Proof:** A sequence of sentences that are immediate consequences of applied inference rules
- **Theorem proving:** process of finding a proof of theorem

Normal forms

Sentences in the propositional logic can be transformed into one of the normal forms. This can simplify the inferences.

Normal forms used:

Conjunctive normal form (CNF)

- conjunction of clauses (clauses include disjunctions of literals)

$$(A \vee B) \wedge (\neg A \vee \neg C \vee D)$$

Disjunctive normal form (DNF)

- Disjunction of terms (terms include conjunction of literals)

$$(A \wedge \neg B) \vee (\neg A \wedge C) \vee (C \wedge \neg D)$$

Conversion to a CNF

Assume: $\neg(A \Rightarrow B) \vee (C \Rightarrow A)$

1. Eliminate $\Rightarrow, \Leftrightarrow$

$$\neg(\neg A \vee B) \vee (\neg C \vee A)$$

2. Reduce the scope of signs through DeMorgan Laws and double negation

$$(A \wedge \neg B) \vee (\neg C \vee A)$$

3. Convert to CNF using the associative and distributive laws

$$(A \vee \neg C \vee A) \wedge (\neg B \vee \neg C \vee A)$$

and

$$(A \vee \neg C) \wedge (\neg B \vee \neg C \vee A)$$

Satisfiability (SAT) problem

Determine whether a sentence in the conjunctive normal form (CNF) is satisfiable (I.e. can evaluate to true)

$$(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg R \vee S) \wedge (\neg P \vee Q \vee \neg T) \ldots$$

It is an instance of a constraint satisfaction problem:

- **Variables:**
  - Propositional symbols (*P, R, T, S*)
  - Values: ***True, False***
- **Constraints:**
  - Every conjunct must evaluate to true, at least one of the literals must evaluate to true
- **All techniques developed for CSPs can be applied to solve the logical inference problem. Why?**

Inference problem and satisfiability

Inference problem:

- we want to show that the sentence α is entailed by KB

Satisfiability:

- The sentence is satisfiable if there is some assignment (interpretation) under which the sentence evaluates to true

Connection:

$KB \models \alpha$ if and only if
$(KB \wedge \neg \alpha)$ is **unsatisfiable**

Consequences:

- inference problem is NP-complete
- programs for solving the SAT problem can be used to solve the inference problem

Universal inference rule: Resolution rule

Sometimes inference rules can be combined into a single rule

Resolution rule

- sound inference rule that works for CNF
- It is complete for **propositional logic (refutation complete)**

$$\frac{A \vee B, \quad \neg A \vee C}{B \vee C}$$

A	B	C	$A \vee B$	$\neg B \vee C$	$A \vee C$
False	*False*	*False*	*False*	*True*	*False*
False	*False*	*True*	*False*	*True*	*True*
False	*True*	*False*	*True*	*False*	*False*
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True	*False*	*False*	*True*	*True*	*True*
True	*False*	*True*	*True*	*True*	*True*
True	*True*	*False*	*True*	*False*	*True*
True	*True*	*True*	*True*	*True*	*True*

Universal rule: Resolution.

Initial obstacle:

- Repeated application of the resolution rule to a KB in CNF may fail to derive new valid sentences

  Example:

  We know: $(A \wedge B)$ We want to show: $(A \vee B)$

  Resolution rule fails to derive it (**incomplete ??)**

A trick to make things work:

- **proof by contradiction**
  - **Disproving:** $KB, \neg \alpha$
  - **Proves the entailment** $KB \models \alpha$

Resolution algorithm

Algorithm:

- **Convert KB to the CNF form;**
- **Apply iteratively the resolution rule** starting from $KB, \neg \alpha$ (in CNF form)
- **Stop when**:
  - Contradiction (empty clause) is reached:
    - $A, \neg A \rightarrow \emptyset$
    - proves entailment.
  - No more new sentences can be derived
    - disproves it.

Example. Resolution.

KB: $(P \wedge Q) \wedge (P \Rightarrow R) \wedge [(Q \wedge R) \Rightarrow S]$ **Theorem:** S

Step 1. convert KB to CNF:

- $P \wedge Q \longrightarrow P \wedge Q$
- $P \Rightarrow R \longrightarrow (\neg P \vee R)$
- $(Q \wedge R) \Rightarrow S \longrightarrow (\neg Q \vee \neg R \vee S)$

KB: $P \quad Q \quad (\neg P \vee R) \quad (\neg Q \vee \neg R \vee S)$

Step 2. Negate the theorem to prove it via refutation

$S \longrightarrow \neg S$

Step 3. Run resolution on the set of clauses

$P \quad Q \quad (\neg P \vee R) \quad (\neg Q \vee \neg R \vee S) \quad \neg S$

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$P \quad Q \quad (\neg P \vee R) \quad (\neg Q \vee \neg R \vee S) \quad \neg S$

Example. Resolution.

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$$R$$

Example. Resolution.

KB: $(P \wedge Q) \wedge (P \Rightarrow R) \wedge [(Q \wedge R) \Rightarrow S]$ **Theorem:** S

$$P \quad Q \quad (\neg P \vee R) \quad (\neg Q \vee \neg R \vee S) \quad \neg S$$

$$R \qquad (\neg R \vee S)$$

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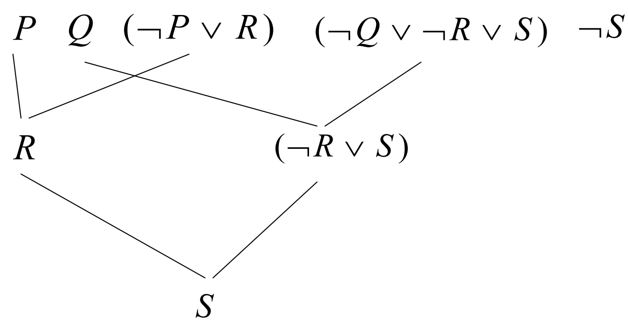

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