CS 1571 Introduction to AI
Lecture 23

Modeling uncertainty using probabilities

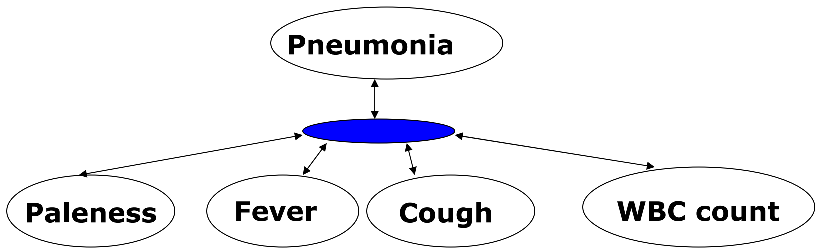
Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Administration

- **Final exam:**
  - December 11, 2006
  - 12:00-1:50pm, 5129 Sennott Square

Uncertainty

To make diagnostic inference possible we need to represent knowledge (axioms) that relate symptoms and diagnosis



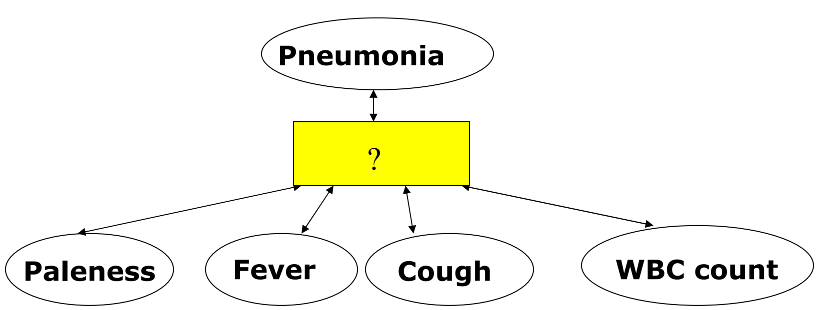
Problem: disease/symptoms relations are not deterministic

- **They are uncertain (or stochastic) and** vary from patient to patient

Modeling the uncertainty.

Key challenges:

- How to represent the relations in the presence of uncertainty?
- How to manipulate such knowledge to make inferences?
  - **Humans can reason with uncertainty.**



Methods for representing uncertainty

Probability theory

- A well defined theory for modeling and reasoning in the presence of uncertainty
- A natural choice to replace certainty factors

Facts (propositional statements)

- Are represented via **random variables** with two or more values

  Example: *Pneumonia* is a random variable

  values: *True* and *False*
- Each value can be achieved **with some probability:**

$$P(Pneumonia = True) = 0.001$$

$$P(WBCcount = high) = 0.005$$

Modeling uncertainty with probabilities

Probabilistic extension of propositional logic.

- **Propositions:**
  - statements about the world
  - Represented by the assignment of values to **random variables**
- **Random variables:**
  - ! **Boolean** $Pneumonia$ is either $True, False$

    Random variable **Values**
  - ! **Multi-valued** $Pain$ is one of $\{Nopain, Mild, Moderate, Severe\}$

    Random variable **Values**
  - **Continuous** $HeartRate$ is a value in $< 0 ; 250 >$

    Random variable **Values**

Probabilities

Unconditional probabilities (prior probabilities)

$P(Pneumonia) = 0.001$ or $P(Pneumonia = True) = 0.001$

$P(Pneumonia = False) = 0.999$

$P(WBCcount = high) = 0.005$

Probability distribution

- Defines probabilities **for all possible value assignments to a random variable**
- Values are mutually exclusive

$P(Pneumonia = True) = 0.001$
$P(Pneumonia = False) = 0.999$

$Pneumonia$	$\mathbf{P}(Pneumonia)$
$True$	0.001
$False$	0.999

Probability distribution

Defines probability for **all possible value assignments**

Example 1:

$P(Pneumonia = True) = 0.001$
$P(Pneumonia = False) = 0.999$

$Pneumonia$	$\mathbf{P}(Pneumonia)$
$True$	0.001
$False$	0.999

$$P(Pneumonia = True) + P(Pneumonia = False) = 1$$

Probabilities sum to 1 !!!

Example 2:

$P(WBCcount = high) = 0.005$
$P(WBCcount = normal) = 0.993$
$P(WBCcount = high) = 0.002$

$WBCcount$	$\mathbf{P}(WBCcount)$
$high$	0.005
$normal$	0.993
low	0.002

Joint probability distribution

Joint probability distribution (for a set variables)

- Defines probabilities for **all possible assignments of values to variables in the set**

Example: variables *Pneumonia* and *WBCcount*

$\mathbf{P}(pneumonia, WBCcount)$

Is represented by 2×3 matrix

WBCcount

Pneumonia		*high*	*normal*	*low*
	True	0.0008	0.0001	0.0001
	False	0.0042	0.9929	0.0019

Joint probabilities

Marginalization

- reduces the dimension of the joint distribution
- Sums variables out

$\mathbf{P}(pneumonia, WBCcount)$ 2×3 matrix

WBCcount

Pneumonia		*high*	*normal*	*low*	$\mathbf{P}(Pneumonia)$
	True	0.0008	0.0001	0.0001	0.001
	False	0.0042	0.9929	0.0019	0.999
$\mathbf{P}(WBCcount)$		0.005	0.993	0.002	

Marginalization (here summing of columns or rows)

Full joint distribution

- **the joint distribution for all variables in the problem**
  – It defines the complete probability model for the problem
- **Example:** pneumonia diagnosis

  Variables: *Pneumonia, Fever, Paleness, WBCcount, Cough*
  – Full joint defines the probability for all possible assignments of values to *Pneumonia, Fever, Paleness, WBCcount, Cough*

$$P(Pneumonia = T, WBCcount = High, Fever = T, Cough = T, Paleness = T)$$
$$P(Pneumonia = T, WBCcount = High, Fever = T, Cough = T, Paleness = F)$$
$$P(Pneumonia = T, WBCcount = High, Fever = T, Cough = F, Paleness = T)$$

… etc

Conditional probabilities

Conditional probability distribution

- Defines probabilities for all possible assignments, given a fixed assignment to some other variable values

  $P(Pneumonia = true \mid WBCcount = high)$

$\mathbf{P}(Pneumonia \mid WBCcount)$ 3 element vector of 2 elements

		WBCcount		
		high	*normal*	*low*
Pneumonia	*True*	0.08	0.0001	0.0001
	False	0.92	0.9999	0.9999
		1.0	1.0	1.0

$$P(Pneumonia = true \mid WBCcount = high)$$
$$+ P(Pneumonia = false \mid WBCcount = high)$$

Conditional probabilities

Conditional probability

- Is defined in terms of the joint probability:

$$P(A \mid B) = \frac{P(A, B)}{P(B)} \text{ s.t. } P(B) \neq 0$$

- **Example:**

$$P(pneumonia = true \mid WBCcount = high) = \frac{P(pneumonia = true, WBCcount = high)}{P(WBCcount = high)}$$

$$P(pneumonia = false \mid WBCcount = high) = \frac{P(pneumonia = false, WBCcount = high)}{P(WBCcount = high)}$$

Conditional probabilities

- **Conditional probability distribution.**

$$P(A \mid B) = \frac{P(A, B)}{P(B)} \text{ s.t. } P(B) \neq 0$$

- **Product rule.** Join probability can be expressed in terms of conditional probabilities

$$P(A, B) = P(A \mid B) P(B)$$

- **Chain rule**. Any joint probability can be expressed as a product of conditionals

$$\begin{aligned} P(X_1, X_2, \ldots X_n) &= P(X_n \mid X_{1,} \ldots X_{n-1}) P(X_{1,} \ldots X_{n-1}) \\ &= P(X_n \mid X_{1,} \ldots X_{n-1}) P(X_{n-1} \mid X_{1,} \ldots X_{n-2}) P(X_{1,} \ldots X_{n-2}) \\ &= \prod_{i=1}^{n} P(X_i \mid X_{1,} \ldots X_{i-1}) \end{aligned}$$

Bayes rule

Conditional probability.

$$P(A \mid B) = \frac{P(A,B)}{P(B)} \qquad P(A,B) = P(B \mid A)P(A)$$

Bayes rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

When is it useful?

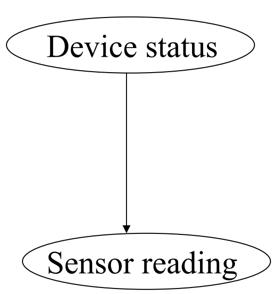
- When we are interested in computing the diagnostic query from the causal probability

$$P(cause \mid effect) = \frac{P(effect \mid cause)P(cause)}{P(effect)}$$

- **Reason:** It is often easier to assess causal probability
  - E.g. Probability of pneumonia causing fever
    vs. probability of pneumonia given fever

Bayes Rule in a simple diagnostic inference.

- **Device** (equipment) operating *normally* or *malfunctioning*.
  - Operation of the device sensed indirectly via a sensor
- **Sensor reading** is either *high* or *low*



P(Device status)

normal	malfunctioning
0.9	0.1

P(Sensor reading| Device status)

Device\Sensor	high	low
normal	0.1	0.9
malfunctioning	0.6	0.4

Bayes Rule in a simple diagnostic inference.

- **Diagnostic inference:** compute the probability of device operating normally or malfunctioning given a sensor reading

$$\mathbf{P}(\text{Device status} \mid \text{Sensor reading} = high) = ?$$

$$= \begin{pmatrix} P(\text{Device status} = normal \mid \text{Sensor reading} = high) \\ P(\text{Device status} = malfunctioning \mid \text{Sensor reading} = high) \end{pmatrix}$$

- Note that typically the opposite conditional probabilities are given to us: they are much easier to estimate
- **Solution:** apply **Bayes rule** to reverse the conditioning variables

Probabilistic inference

Various inference tasks:

- **Diagnostic task. (from effect to cause)**

$$\mathbf{P}(Pneumonia \mid Fever = T)$$

- **Prediction task. (from cause to effect)**

$$\mathbf{P}(Fever \mid Pneumonia = T)$$

- **Other probabilistic queries** (queries on joint distributions).

$$\mathbf{P}(Fever)$$

$$\mathbf{P}(Fever, ChestPain)$$

Inference

Any query can be computed from the full joint distribution !!!

- **Joint over a subset of variables** is obtained through marginalization

$$P(A=a, C=c) = \sum_i \sum_j P(A=a, B=b_i, C=c, D=d_j)$$

- **Conditional probability over set of variables**, given other variables' values is obtained through marginalization and definition of conditionals

$$P(D=d \mid A=a, C=c) = \frac{P(A=a, C=c, D=d)}{P(A=a, C=c)}$$

$$= \frac{\sum_i P(A=a, B=b_i, C=c, D=d)}{\sum_i \sum_j P(A=a, B=b_i, C=c, D=d_j)}$$

Inference

Any query can be computed from the full joint distribution !!!

- Any joint probability can be expressed as a product of conditionals via the **chain rule**.

$$P(X_1, X_2, \ldots X_n) = P(X_n \mid X_1, \ldots X_{n-1}) P(X_1, \ldots X_{n-1})$$

$$= P(X_n \mid X_1, \ldots X_{n-1}) P(X_{n-1} \mid X_1, \ldots X_{n-2}) P(X_1, \ldots X_{n-2})$$

$$= \prod_{i=1}^{n} P(X_i \mid X_1, \ldots X_{i-1})$$

- Sometimes it is easier to define the distribution in terms of conditional probabilities:
  - E.g. $\mathbf{P}(Fever \mid Pneumonia = T)$
    $\mathbf{P}(Fever \mid Pneumonia = F)$

Modeling uncertainty with probabilities

- Defining the **full joint distribution** makes it possible to represent and reason with uncertainty in a uniform way
- We are able to handle an arbitrary inference problem

Problems:

- **Space complexity.** To store a full joint distribution we need to remember $O(d^n)$ numbers.

  n – number of random variables, d – number of values
- **Inference (time) complexity.** To compute some queries requires $O(d^n)$ steps.
- **Acquisition problem.** Who is going to define all of the probability entries?

Medical diagnosis example

- **Space complexity**.
  - Pneumonia (2 values: T,F), Fever (2: T,F), Cough (2: T,F), WBCcount (3: high, normal, low), paleness (2: T,F)
  - Number of assignments: 2*2*2*3*2=48
  - We need to define at least 47 probabilities.
- **Time complexity**.
  - Assume we need to compute the marginal of Pneumonia=T from the full joint

$$P(Pneumonia = T) =$$
$$= \sum_{i \in T,F} \sum_{j \in T,F} \sum_{k=h,n,l} \sum_{u \in T,F} P(Fever = i, Cough = j, WBCcount = k, Pale = u)$$

  - Sum over: 2*2*3*2=24 combinations

Modeling uncertainty with probabilities

- **Knowledge based system era (70s – early 80's)**
  - **Extensional non-probabilistic models**
  - Solve the space, time and acquisition bottlenecks in probability-based models
  - froze the development and advancement of KB systems and contributed to the slow-down of AI in 80s in general

- Breakthrough (late 80s, beginning of 90s)
  - **Bayesian belief networks**
    - Give solutions to the space, acquisition bottlenecks
    - Partial solutions for time complexities
- Bayesian belief network

Bayesian belief networks (BBNs)

Bayesian belief networks.

- Represent the full joint distribution over the variables more compactly with a **smaller number of parameters**.
- Take advantage of **conditional and marginal independences** among random variables

- **A and B are independent**

$$P(A,B) = P(A)P(B)$$

- **A and B are conditionally independent given C**

$$P(A,B \mid C) = P(A \mid C)P(B \mid C)$$

$$P(A \mid C,B) = P(A \mid C)$$