CS 1571 Introduction to AI
Lecture 16

Propositional logic.

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Logical inference problem

Logical inference problem:

- **Given:**
  - a knowledge base KB (a set of sentences) and
  - a sentence α (called **a theorem**),
- **Does a KB semantically entail** α ? $KB \overset{?}{\models} \alpha$

  In other words: In all interpretations in which sentences in the KB are true, is also α true?

Approaches:

- **Truth-table approach**
- **Inference rules**
- **Conversion to SAT**
  - **Resolution refutation**

Truth-table approach

Problem: $KB \models \alpha$?

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth tables:

- enumerate truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols) and check

Example:

		KB		α	
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$	
True	*True*	*True*	*True*	*True*	✓
True	*False*	*True*	*False*	*False*	
False	*True*	*True*	*False*	*False*	
False	*False*	*False*	*True*	*False*	

Inference rules approach.

Motivation: **we do not want to blindly generate and check all interpretations !!!**

Inference rules:

- Represent **sound inference patterns** repeated in inferences
- Application of many inference rules allows us to infer new sound conclusions and hence prove theorems

- **An example of an inference rule: Modus ponens**

$$\frac{A \Rightarrow B, \quad A}{B}$$

← premise

← conclusion

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P **From 1 and And-elim**
5. R **From 2,4 and Modus ponens**
6. Q **From 1 and And-elim**
7. $(Q \wedge R)$ **From 5,6 and And-introduction**
8. S **From 7,3 and Modus ponens**

Proved: S

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Nondeterministic steps

4. P **From 1 and And-elim**
5. R **From 2,4 and Modus ponens**
6. Q **From 1 and And-elim**
7. $(Q \wedge R)$ **From 5,6 and And-introduction**
8. S **From 7,3 and Modus ponens**

Proved: S

Logic inferences and search

Inference rule method as a search problem:

- **State:** a set of sentences that are known to be true
- **Initial state**: a set of sentences in the KB
- **Operators**: applications of inference rules
  - Allow us to add new sound sentences to old ones
- **Goal state:** a theorem α is derived from KB

Logic inference:

- **Proof:** A sequence of sentences that are immediate consequences of applied inference rules
- **Theorem proving:** process of finding a proof of theorem

Satisfiability (SAT) problem

Determine whether a sentence in the conjunctive normal form (CNF) is satisfiable (I.e. can evaluate to true)

$$(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg R \vee S) \wedge (\neg P \vee Q \vee \neg T) \ldots$$

It is an instance of a constraint satisfaction problem:

- **Variables:**
  - Propositional symbols (*P, R, T, S*)
  - Values: ***True, False***
- **Constraints:**
  - Every conjunct must evaluate to true, at least one of the literals must evaluate to true
- **All techniques developed for CSPs can be applied to solve the logical inference problem. Why?**

Inference problem and satisfiability

Inference problem:

- we want to show that the sentence α is entailed by KB

Satisfiability:

- The sentence is satisfiable if there is some assignment (interpretation) under which the sentence evaluates to true

Connection:

$KB \models \alpha$ if and only if

$(KB \wedge \neg \alpha)$ is **unsatisfiable**

Consequences:

- inference problem is NP-complete
- programs for solving the SAT problem can be used to solve the inference problem (Simulated-annealing, WALKSAT)

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- **Variables:**
  - Propositional symbols (*P, R, T, S*)
  - Values: ***True, False***
- **Constraints:**
  - Every conjunct must evaluate to true, at least one of the literals must evaluate to true
- **Why is this important?** All techniques developed for CSPs can be applied to solve the logical inference problem !!

Resolution algorithm

Algorithm:

1. **Convert KB to the CNF form;**
2. **Apply iteratively the resolution rule** starting from $KB, \neg \alpha$ (in the CNF form)
3. **Stop when**:
   - Contradiction (empty clause) is reached:
     - $A, \neg A \rightarrow \emptyset$
     - **proves the entailment.**
   - No more new sentences can be derived
     - **Rejects (disproves) the entailment**.

Example. Resolution.

KB: $(P \wedge Q) \wedge (P \Rightarrow R) \wedge [(Q \wedge R) \Rightarrow S]$ **Theorem:** S

Example. Resolution.

KB: $(P \wedge Q) \wedge (P \Rightarrow R) \wedge [(Q \wedge R) \Rightarrow S]$ **Theorem:** S

Step 1. convert KB to CNF:

- $P \wedge Q \longrightarrow P \wedge Q$
- $P \Rightarrow R \longrightarrow (\neg P \vee R)$
- $(Q \wedge R) \Rightarrow S \longrightarrow (\neg Q \vee \neg R \vee S)$

KB: $P \quad Q \quad (\neg P \vee R) \quad (\neg Q \vee \neg R \vee S)$

Step 2. Negate the theorem to prove it via refutation

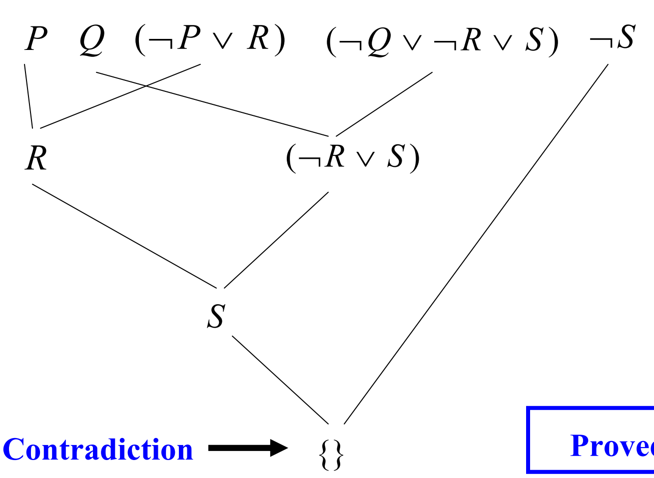
$S \longrightarrow \neg S$

Step 3. Run resolution on the set of clauses

$P \quad Q \quad (\neg P \vee R) \quad (\neg Q \vee \neg R \vee S) \quad \neg S$

Example. Resolution.

KB: $(P \wedge Q) \wedge (P \Rightarrow R) \wedge [(Q \wedge R) \Rightarrow S]$ **Theorem:** S



Contradiction $\longrightarrow$ {}

Proved: S

KB in restricted forms

- If the sentences in the KB are restricted to some special forms other sound inference rules may become complete

Example:

- **Horn form (Horn normal form)**

$$(A \vee \neg B) \wedge (\neg A \vee \neg C \vee D)$$

Can be written also as: $(B \Rightarrow A) \wedge ((A \wedge C) \Rightarrow D)$

- **Modus ponens:**
  - is the "universal "(complete) rule for the sentences in the Horn form

$$\frac{A \Rightarrow B, \quad A}{B} \qquad \frac{A_1 \wedge A_2 \wedge \ldots \wedge A_k \Rightarrow B, A_1, A_2, \ldots A_k}{B}$$

KB in Horn form

- **Horn form:** a clause with at most one positive literal

$$(A \vee \neg B) \wedge (\neg A \vee \neg C \vee D)$$

- **Not all sentences in propositional logic can be converted into the Horn form**
- **KB in Horn normal form**:
  - Two types of propositional statements:
    - Implications: called **rules** $(B \Rightarrow A)$
    - Propositional symbols: **facts** B
- **Application of the modus ponens:**
  - Infers new facts from previous facts

$$\frac{A \Rightarrow B, \quad A}{B}$$

Forward and backward chaining

Two inference procedures based on **modus ponens** for **Horn KBs**:

- **Forward chaining**

  Idea: Whenever the premises of a rule are satisfied, infer the conclusion. Continue with rules that became satisfied.

- **Backward chaining (goal reduction)**

  Idea: To prove the fact that appears in the conclusion of a rule prove the premises of the rule. Continue recursively.

Both procedures are **complete for KBs in the Horn form** !!!

Forward chaining example

- **Forward chaining**

  Idea: Whenever the premises of a rule are satisfied, infer the conclusion. Continue with rules that became satisfied.

Assume the KB with the following rules and facts:

KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

R3: $C \wedge F \Rightarrow G$

F1: A
F2: B
F3: D

Theorem: E ?

Forward chaining example

Theorem: E

KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

R3: $C \wedge F \Rightarrow G$

F1: A

F2: B

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Forward chaining example

Theorem: E

KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

R3: $C \wedge F \Rightarrow G$

F1: A

F2: B

F3: D

Rule R1 is satisfied.

F4: C

Forward chaining example

Theorem: E

KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

R3: $C \wedge F \Rightarrow G$

F1: A
F2: B
F3: D

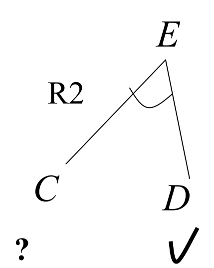
Rule R1 is satisfied.

F4: C

Rule R2 is satisfied.

F5: E ✓

Backward chaining example



KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

R3: $C \wedge F \Rightarrow G$

F1: A
F2: B
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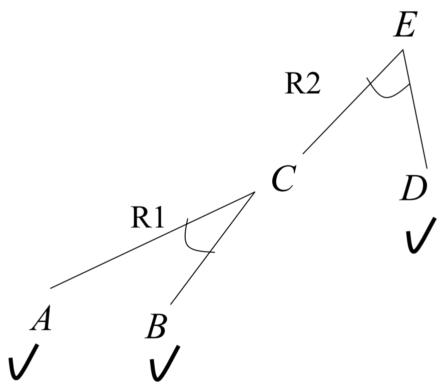
- Backward chaining is more focused:
  - tries to prove the theorem only

Backward chaining example



KB: R1: $A \wedge B \Rightarrow C$

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F1: A

F2: B

F3: D

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  - tries to prove the theorem only

KB agents based on propositional logic

- Propositional logic allows us to build **knowledge-based agents** capable of answering queries about the world by infering new facts from the known ones
- **Example:** an agent for diagnosis of a bacterial disease

Facts: The stain of the organism is gram-positive
The growth conformation of the organism is chains

Rules: **(If)** The stain of the organism is gram-positive $\wedge$
The morphology of the organism is coccus $\wedge$
The growth conformation of the organism is chains
(Then) $\Rightarrow$ The identity of the organism is streptococcus