CS 1571 Introduction to AI
Lecture 15

Propositional logic

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Announcements

Midterm exam:
- Wednesday, October 25, 2006

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs1571/

Logical inference problem

Logical inference problem:

- **Given:**
  - a knowledge base KB (a set of sentences) and
  - a sentence α (called **a theorem**),
- **Does a KB semantically entail** α ? $KB \models \alpha$?

In other words: In all interpretations in which sentences in the KB are true, is also α true?

Question: Is there a procedure (program) that can decide this problem in a finite number of steps?

Answer: **Yes**. Logical inference problem for the propositional logic is **decidable**.

Solving logical inference problem

In the following:

How to design the procedure that answers:

$$KB \models \alpha \ ?$$

Three approaches:

- **Truth-table approach**
- **Inference rules**
- **Conversion to the inverse SAT problem**
  - **Resolution-refutation**

Truth-table approach

Problem: $KB \models \alpha$?

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False values to propositional symbols)

Example:

		KB		α
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$
True	*True*	*True*	*True*	*True*
True	*False*	*True*	*False*	*False*
False	*True*	*True*	*False*	*False*
False	*False*	*False*	*True*	*False*

Truth-table approach

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True	*True*	*True*	*True*	*True*
True	*False*	*True*	*False*	*False*
False	*True*	*True*	*False*	*False*
False	*False*	*False*	*True*	*False*

Truth-table approach

Problem: $KB \models \alpha$?

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Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

Example:

		KB		α	
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$	
True	*True*	*True*	*True*	*True*	✓
True	*False*	*True*	*False*	*False*	
False	*True*	*True*	*False*	*False*	
False	*False*	*False*	*True*	*False*	

Inference rules approach.

$KB \models \alpha$?

Problem with the truth table approach:

- the truth table is **exponential** in the number of propositional symbols (we checked all assignments)
- KB is true on only a smaller subset

How to make the process more efficient?

Solution: check only entries for which KB is *True*.

This is the idea behind the inference rules approach

Inference rules:

- Represent sound inference patterns repeated in inferences
- Can be used to generate new (sound) sentences from the existing ones

Inference rules for logic

- **Modus ponens**

$$\frac{A \Rightarrow B, \quad A}{B}$$

← **premise**

← **conclusion**

- If both sentences in the premise are true then conclusion is true.
- The modus ponens inference rule is **sound.**
  - We can prove this through the truth table.

A	B	$A \Rightarrow B$
False	*False*	*True*
False	*True*	*True*
True	*False*	*False*
True	$\boxed{\textit{True}}$	*True*

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P

From 1 and And-elim

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R

From 2,4 and Modus ponens

$$\frac{A \Rightarrow B, \quad A}{B}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q

From 1 and And-elim

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q
7. $(Q \wedge R)$

From 5,6 and And-introduction

$$\frac{A_1, A_2, \quad A_n}{A_1 \wedge A_2 \wedge \quad A_n}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q
7. $(Q \wedge R)$
8. S **From 7,3 and Modus ponens**

$$\frac{A \Rightarrow B, \quad A}{B}$$

Proved: S

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P **From 1 and And-elim**
5. R **From 2,4 and Modus ponens**
6. Q **From 1 and And-elim**
7. $(Q \wedge R)$ **From 5,6 and And-introduction**
8. S **From 7,3 and Modus ponens**

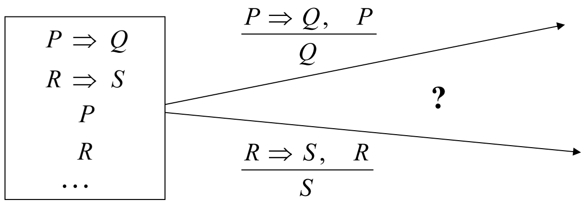
Proved: S

Logic inferences and search

- To show that theorem α holds for a KB
  - we may need to apply a number of sound inference rules

Problem: many possible rules to can be applied next

Looks familiar?



This is an instance of a search problem:

Truth table method (from the search perspective):

- blind enumeration and checking

Logic inferences and search

Inference rule method as a search problem:

- **State:** a set of sentences that are known to be true
- **Initial state**: a set of sentences in the KB
- **Operators**: applications of inference rules
  - Allow us to add new sound sentences to old ones
- **Goal state:** a theorem α is derived from KB

Logic inference:

- **Proof:** A sequence of sentences that are immediate consequences of applied inference rules
- **Theorem proving:** process of finding a proof of theorem

Normal forms

Sentences in the propositional logic can be transformed into one of the normal forms. This can simplify the inferences.

Normal forms used:

Conjunctive normal form (CNF)

- conjunction of clauses (clauses include disjunctions of literals)

$$(A \vee B) \wedge (\neg A \vee \neg C \vee D)$$

Disjunctive normal form (DNF)

- Disjunction of terms (terms include conjunction of literals)

$$(A \wedge \neg B) \vee (\neg A \wedge C) \vee (C \wedge \neg D)$$

Conversion to a CNF

Assume: $\neg(A \Rightarrow B) \vee (C \Rightarrow A)$

1. Eliminate $\Rightarrow, \Leftrightarrow$

$$\neg(\neg A \vee B) \vee (\neg C \vee A)$$

2. Reduce the scope of signs through DeMorgan Laws and double negation

$$(A \wedge \neg B) \vee (\neg C \vee A)$$

3. Convert to CNF using the associative and distributive laws

$$(A \vee \neg C \vee A) \wedge (\neg B \vee \neg C \vee A)$$

and

$$(A \vee \neg C) \wedge (\neg B \vee \neg C \vee A)$$

Satisfiability (SAT) problem

Determine whether a sentence in the conjunctive normal form (CNF) is satisfiable (I.e. can evaluate to true)

$$(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg R \vee S) \wedge (\neg P \vee Q \vee \neg T) \ldots$$

It is an instance of a constraint satisfaction problem:

- **Variables:**
  - Propositional symbols (*P, R, T, S*)
  - Values: ***True, False***
- **Constraints:**
  - Every conjunct must evaluate to true, at least one of the literals must evaluate to true
- **All techniques developed for CSPs can be applied to solve the logical inference problem. Why?**

Inference problem and satisfiability

Inference problem:

- we want to show that the sentence α is entailed by KB

Satisfiability:

- The sentence is satisfiable if there is some assignment (interpretation) under which the sentence evaluates to true

Connection:

$KB \models \alpha$ if and only if

$(KB \wedge \neg \alpha)$ is **unsatisfiable**

Consequences:

- inference problem is NP-complete
- programs for solving the SAT problem can be used to solve the inference problem

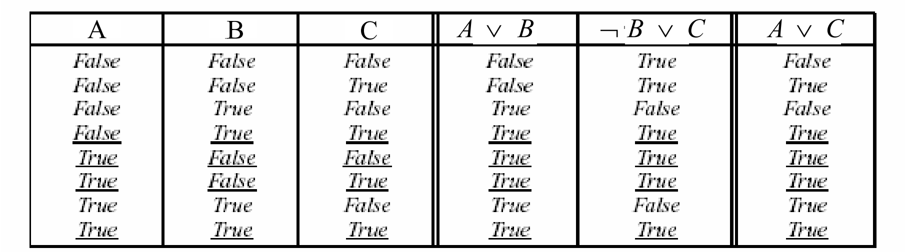
Universal inference rule: Resolution rule

Sometimes inference rules can be combined into a single rule

Resolution rule

- sound inference rule that works for CNF
- It is complete for **propositional logic (refutation complete)**

$$\frac{A \vee B, \quad \neg A \vee C}{B \vee C}$$



A	B	C	$A \vee B$	$\neg B \vee C$	$A \vee C$
False	False	False	False	True	False
False	False	True	False	True	True
False	True	False	True	False	False
False	True	True	True	True	True
True	False	False	True	True	True
True	False	True	True	True	True
True	True	False	True	False	True
True	True	True	True	True	True

Universal rule: Resolution.

Initial obstacle:

- Repeated application of the resolution rule to a KB in CNF may fail to derive new valid sentences

  Example:

  We know: $(A \wedge B)$ We want to show: $(A \vee B)$

  Resolution rule fails to derive it (**incomplete ??)**

A trick to make things work:

- **proof by contradiction**
  - **Disproving:** $KB, \neg \alpha$
  - **Proves the entailment** $KB \models \alpha$

Resolution algorithm

Algorithm:

- **Convert KB to the CNF form;**
- **Apply iteratively the resolution rule** starting from
  $KB, \neg \alpha$ (in CNF form)
- **Stop when**:
  - Contradiction (empty clause) is reached:
    - $A, \neg A \rightarrow \emptyset$
    - proves entailment.
  - No more new sentences can be derived
    - disproves it.

Example. Resolution.

KB: $(P \wedge Q) \wedge (P \Rightarrow R) \wedge [(Q \wedge R) \Rightarrow S]$ **Theorem:** S

Step 1. convert KB to CNF:

- $P \wedge Q \longrightarrow P \wedge Q$
- $P \Rightarrow R \longrightarrow (\neg P \vee R)$
- $(Q \wedge R) \Rightarrow S \longrightarrow (\neg Q \vee \neg R \vee S)$

KB: $P \quad Q \quad (\neg P \vee R) \quad (\neg Q \vee \neg R \vee S)$

Step 2. Negate the theorem to prove it via refutation

$S \longrightarrow \neg S$

Step 3. Run resolution on the set of clauses

$P \quad Q \quad (\neg P \vee R) \quad (\neg Q \vee \neg R \vee S) \quad \neg S$

Example. Resolution.

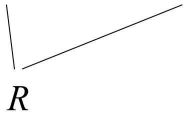
KB: $(P \wedge Q) \wedge (P \Rightarrow R) \wedge [(Q \wedge R) \Rightarrow S]$ **Theorem:** S

$P \quad Q \quad (\neg P \vee R) \quad (\neg Q \vee \neg R \vee S) \quad \neg S$

Example. Resolution.

KB: $(P \wedge Q) \wedge (P \Rightarrow R) \wedge [(Q \wedge R) \Rightarrow S]$ **Theorem:** S

$P \quad Q \quad (\neg P \vee R) \quad (\neg Q \vee \neg R \vee S) \quad \neg S$



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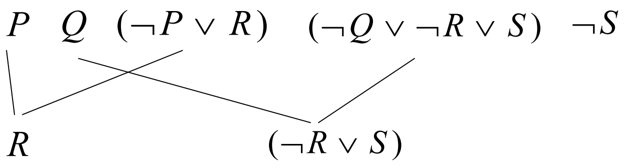
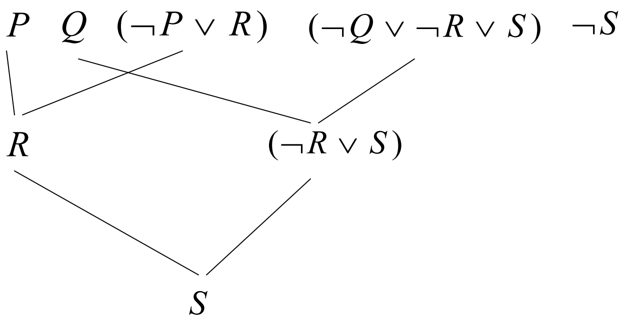


P Q $(\neg P \vee R)$ $(\neg Q \vee \neg R \vee S)$ $\neg S$

R $(\neg R \vee S)$

Example. Resolution.

KB: $(P \wedge Q) \wedge (P \Rightarrow R) \wedge [(Q \wedge R) \Rightarrow S]$ **Theorem:** S



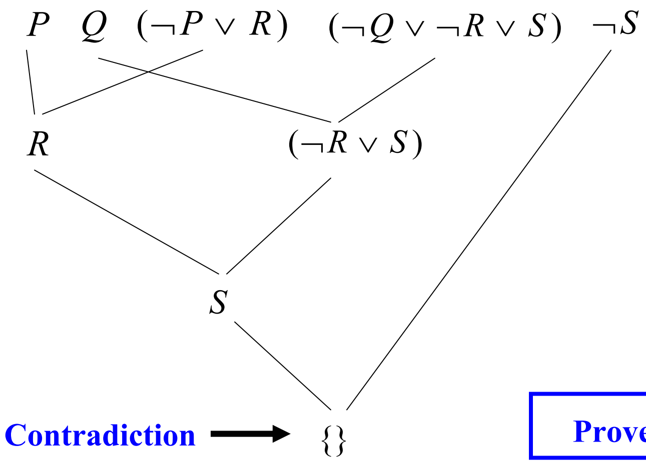
P Q $(\neg P \vee R)$ $(\neg Q \vee \neg R \vee S)$ $\neg S$

R $(\neg R \vee S)$

S

Example. Resolution.

KB: $(P \wedge Q) \wedge (P \Rightarrow R) \wedge [(Q \wedge R) \Rightarrow S]$ **Theorem:** S



P Q $(\neg P \vee R)$ $(\neg Q \vee \neg R \vee S)$ $\neg S$

R $(\neg R \vee S)$

S

Contradiction $\longrightarrow$ $\{\}$

Proved: S