# CS 1571 Introduction to AI
# Lecture 14

# Propositional logic II

**Milos Hauskrecht**
milos@cs.pitt.edu
5329 Sennott Square



## Announcements

- **Homework assignment 5 is out**
  - **Deadline extended till Monday, October 23**
  - **Tree.cpp now implements 'stochastic' minimax**
- **No new homework assignment**
- **Midterm exam:**
  - **Wednesday, October 25, 2006**

**Course web page:**

http://www.cs.pitt.edu/~milos/courses/cs1571/

## Knowledge-based agent

| Knowledge base |
|---|
| Inference engine |

- **Knowledge base (KB):**
  - A set of sentences that describe facts about the world in some formal (representational) language
  - **Domain specific**
- **Inference engine:**
  - A set of procedures that use the representational language to infer new facts from known ones or answer a variety of KB queries. Inferences typically require search.
  - **Domain independent**



## Knowledge representation

- The objective of knowledge representation is to express the knowledge about the world in a computer-tractable form
- Key aspects of knowledge representation languages:
  - **Syntax:** describes how sentences are formed in the language
  - **Semantics:** describes the meaning of sentences, what is it the sentence refers to in the real world
  - **Computational aspect:** describes how sentences and objects are manipulated in concordance with semantical conventions

**Many KB systems rely on some variant of logic**

# Logic

A formal language for expressing knowledge and ways of reasoning.

**Logic** is defined by**:**

- **A set of sentences**
  - A sentence is constructed from a set of primitives according to syntax rules.
- **A set of interpretations**
  - An interpretation gives a semantic to primitives. It associates primitives with values.
- **The valuation (meaning) function** $V$
  - Assigns a value (typically the truth value) to a given sentence under some interpretation

$$V : \text{sentence} \times \text{interpretation} \rightarrow \{True, False\}$$



# Propositional logic. Syntax

- **Formally propositional logic P:**
  - Is defined **by Syntax+interpretation+semantics of P**

**Syntax:**

- **Symbols (alphabet)** in P:
  - **Constants**: ***True, False***
  - **Propositional symbols**

    Examples:
    - *P*
    - *Pitt is located in the Oakland section of Pittsburgh.,*
    - *It rains outside,* etc.
  - **A set of connectives**:

$$\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$$

## Propositional logic. Syntax

**Sentences in the propositional logic:**

- **Atomic sentences:**
  - **Constructed from constants and propositional symbols**
  - True, False are (atomic) sentences
  - $P, Q$ or *Light in the room is on, It rains outside* are (atomic) sentences
- **Composite sentences:**
  - **Constructed from valid sentences via connectives**
  - If $A, B$ are sentences then

    $\neg A \quad (A \wedge B) \quad (A \vee B) \quad (A \Rightarrow B) \quad (A \Leftrightarrow B)$

    or $(A \vee B) \wedge (A \vee \neg B)$

    are sentences



## Propositional logic. Semantics.

**The semantic gives the meaning to sentences.**

the semantics in the propositional logic is defined by:

1. **Interpretation of propositional symbols and constants**
   - Semantics of atomic sentences
2. **Through the meaning of connectives**
   - Meaning (semantics) of composite sentences

## Semantic: propositional symbols

A **propositional symbol**

- a statement about the world that is either true or false

  Examples:
  - *Pitt is located in the Oakland section of Pittsburgh*
  - *It rains outside*
  - *Light in the room is on*
- An **interpretation** maps symbols to one of the two values: ***True (T),*** or ***False (F)***, depending on whether the symbol is satisfied in the world

  **I**: *Light in the room is on ->* ***True,*** *It rains outside ->* ***False***

  **I'**: *Light in the room is on ->* ***False,*** *It rains outside ->* ***False***



## Semantic: propositional symbols

The **meaning (value)** of the propositional symbol for a specific interpretation is given by its interpretation

**I**: *Light in the room is on ->* ***True,*** *It rains outside ->* ***False***

*V(Light in the room is on,* **I***) =* ***True***

*V( It rains outside,* **I***) =* ***False***

**I'**: *Light in the room is on ->* ***False,*** *It rains outside ->* ***False***

*V(Light in the room is on,* **I'***) =* ***False***

## Semantics: constants

- **The meaning (truth) of constants:**
  - True and False constants are always (under any interpretation) assigned the corresponding ***True,False*** value

$$\left. \begin{array}{l} V(True, \mathbf{I}) = \boldsymbol{True} \\ V(False, \mathbf{I}) = \boldsymbol{False} \end{array} \right\} \quad \text{For any interpretation } \mathbf{I}$$



## Semantics: composite sentences.

- **The meaning (truth value) of complex propositional sentences.**
  - Determined using the standard rules of logic:

| $P$ | $Q$ | $\neg P$ | $P \wedge Q$ | $P \vee Q$ | $P \Rightarrow Q$ | $P \Leftrightarrow Q$ |
|---|---|---|---|---|---|---|
| ***True*** | ***True*** | ***False*** | ***True*** | ***True*** | ***True*** | ***True*** |
| ***True*** | ***False*** | ***False*** | ***False*** | ***True*** | ***False*** | ***False*** |
| ***False*** | ***True*** | ***True*** | ***False*** | ***True*** | ***True*** | ***False*** |
| ***False*** | ***False*** | ***True*** | ***False*** | ***False*** | ***True*** | ***True*** |

# Model, validity and satisfiability

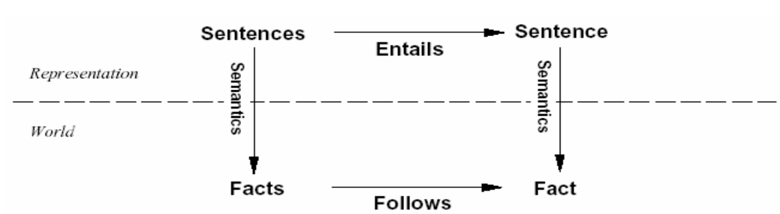
- A **model (in logic):** An interpretation is a model for a set of sentences if it assigns true to each sentence in the set.
- A sentence is **satisfiable** if it has a model;
  – There is at least one interpretation under which the sentence can evaluate to True.
- A sentence is **valid** if it is ***True*** in all interpretations
  – i.e., if its negation is **not satisfiable** (leads to contradiction)

| $P$ | $Q$ | $P \vee Q$ | $(P \vee Q) \wedge \neg Q$ | $((P \vee Q) \wedge \neg Q) \Rightarrow P$ |
|---|---|---|---|---|
| ***True*** | ***True*** | ***True*** | ***False*** | ***True*** |
| ***True*** | ***False*** | ***True*** | ***True*** | ***True*** |
| ***False*** | ***True*** | ***True*** | ***False*** | ***True*** |
| ***False*** | ***False*** | ***False*** | ***False*** | ***True*** |



# Entailment

- **Entailment** reflects the relation of one fact in the world following from the others




- Entailment $KB \models \alpha$
- Knowledge base KB entails sentence $\alpha$ if and only if
  $\alpha$ is true in all worlds where KB is true

# Inference.

**Inference** is a process by which conclusions are reached.

- We want to implement the inference process on a computer !!

Assume an **inference procedure *i*** that

- derives a sentence $\alpha$ from the KB : $KB \vdash_i \alpha$

**Properties of the inference procedure in terms of entailment**

- **Soundness:** An inference procedure is **sound**

  If $KB \vdash_i \alpha$ then it is true that $KB \models \alpha$

- **Completeness:** An inference procedure is **complete**

  If $KB \models \alpha$ then it is true that $KB \vdash_i \alpha$



# Logical inference problem

**Logical inference problem**:

- **Given:**
  - a knowledge base KB (a set of sentences) and
  - a sentence $\alpha$ (called **a theorem**),
- **Does a KB semantically entail** $\alpha$ ? $KB \models \alpha$ ?

In other words: In all interpretations in which sentences in the KB are true, is also $\alpha$ true?

**Question:** Is there a procedure (program) that can decide this problem in a finite number of steps?

**Answer:** **Yes**. Logical inference problem for the propositional logic is **decidable**.

## Solving logical inference problem

In the following:

**How to design the procedure that answers:**

$$KB \models \alpha \ ?$$

**Three approaches:**

- **Truth-table approach**
- **Inference rules**
- **Conversion to the inverse SAT problem**
  - **Resolution-refutation**



## Truth-table approach

**Problem:** $KB \models \alpha \ ?$

- We need to check all possible interpretations for which the KB is true (models of KB) whether $\alpha$ is true for each of them

**Truth table**:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False values to propositional symbols)

**Example:**

| | | KB | | $\alpha$ |
|---|---|---|---|---|
| $P$ | $Q$ | $P \vee Q$ | $P \Leftrightarrow Q$ | $(P \vee \neg Q) \wedge Q$ |
| *True* | *True* | *True* | *True* | *True* |
| *True* | *False* | *True* | *False* | *False* |
| *False* | *True* | *True* | *False* | *False* |
| *False* | *False* | *False* | *True* | *False* |

# Truth-table approach

**Problem:** $KB \models \alpha$ ?

- We need to check all possible interpretations for which the KB is true (models of KB) whether $\alpha$ is true for each of them

**Truth table:**

- enumerates truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

**Example:**

| | | KB | | $\alpha$ |
|---|---|---|---|---|
| $P$ | $Q$ | $P \vee Q$ | $P \Leftrightarrow Q$ | $(P \vee \neg Q) \wedge Q$ |
| *True* | *True* | *True* | *True* | *True* |
| *True* | *False* | *True* | *False* | *False* |
| *False* | *True* | *True* | *False* | *False* |
| *False* | *False* | *False* | *True* | *False* |



# Truth-table approach

**Problem:** $KB \models \alpha$ ?

- We need to check all possible interpretations for which the KB is true (models of KB) whether $\alpha$ is true for each of them

**Truth table:**

- enumerates truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

**Example:**

| | | KB | | $\alpha$ | |
|---|---|---|---|---|---|
| $P$ | $Q$ | $P \vee Q$ | $P \Leftrightarrow Q$ | $(P \vee \neg Q) \wedge Q$ | |
| *True* | *True* | *True* | *True* | *True* | ✓ |
| *True* | *False* | *True* | *False* | *False* | |
| *False* | *True* | *True* | *False* | *False* | |
| *False* | *False* | *False* | *True* | *False* | |

## Truth-table approach

**A two steps procedure:**

1. **Generate table for all possible interpretations**
2. Check whether the sentence $\alpha$ evaluates to true whenever $KB$ evaluates to true

**Example**: $KB = (A \vee C) \wedge (B \vee \neg C)$   $\alpha = (A \vee B)$

| $A$ | $B$ | $C$ | $A \vee C$ | $(B \vee \neg C)$ | $KB$ | $\alpha$ |
|---|---|---|---|---|---|---|
| *True* | *True* | *True* | | | | |
| *True* | *True* | *False* | | | | |
| *True* | *False* | *True* | | | | |
| *True* | *False* | *False* | | | | |
| *False* | *True* | *True* | | | | |
| *False* | *True* | *False* | | | | |
| *False* | *False* | *True* | | | | |
| *False* | *False* | *False* | | | | |



## Truth-table approach

**A two steps procedure:**

1. Generate table for all possible interpretations
2. **Check whether the sentence $\alpha$ evaluates to true whenever $KB$ evaluates to true**

**Example**: $KB = (A \vee C) \wedge (B \vee \neg C)$   $\alpha = (A \vee B)$

| $A$ | $B$ | $C$ | $A \vee C$ | $(B \vee \neg C)$ | $KB$ | $\alpha$ |
|---|---|---|---|---|---|---|
| *True* | *True* | *True* | *True* | *True* | *True* | *True* |
| *True* | *True* | *False* | *True* | *True* | *True* | *True* |
| *True* | *False* | *True* | *True* | *False* | *False* | *True* |
| *True* | *False* | *False* | *True* | *True* | *True* | *True* |
| *False* | *True* | *True* | *True* | *True* | *True* | *True* |
| *False* | *True* | *False* | *False* | *True* | *False* | *True* |
| *False* | *False* | *True* | *True* | *False* | *False* | *False* |
| *False* | *False* | *False* | *False* | *True* | *False* | *False* |

## Truth-table approach

**A two steps procedure:**

1. Generate table for all possible interpretations
2. **Check whether the sentence $\alpha$ evaluates to true whenever *KB* evaluates to true**

**Example**: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

| $A$ | $B$ | $C$ | $A \vee C$ | $(B \vee \neg C)$ | $KB$ | $\alpha$ | |
|---|---|---|---|---|---|---|---|
| True | True | True | True | True | True | True | ✓ |
| True | True | False | True | True | True | True | ✓ |
| True | False | True | True | False | False | True | |
| True | False | False | True | True | True | True | ✓ |
| False | True | True | True | True | True | True | ✓ |
| False | True | False | False | True | False | True | |
| False | False | True | True | False | False | False | |
| False | False | False | False | True | False | False | |



## Truth-table approach

$KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

| $A$ | $B$ | $C$ | $A \vee C$ | $(B \vee \neg C)$ | $KB$ | $\alpha$ |
|---|---|---|---|---|---|---|
| True | True | True | True | True | True | True |
| True | True | False | True | True | True | True |
| True | False | True | True | False | False | True |
| True | False | False | True | True | True | True |
| False | True | True | True | True | True | True |
| False | True | False | False | True | False | True |
| False | False | True | True | False | False | False |
| False | False | False | False | True | False | False |

KB entails $\alpha$

- The **truth-table approach** is **sound and complete** for the propositional logic!!

## Limitations of the truth table approach.

$$KB \models \alpha \ ?$$

**What is the computational complexity of the truth table approach?**

- **?**



## Limitations of the truth table approach.

$$KB \models \alpha \ ?$$

**What is the computational complexity of the truth table approach?**

**Exponential in the number of proposition symbols**

$2^n$ Rows in the table has to be filled

## Limitations of the truth table approach.

$$KB \models \alpha \ ?$$

**What is the computational complexity of the truth table approach?**

**Exponential in the number of the proposition symbols**

$2^n$ Rows in the table has to be filled

**But typically only for a small subset of rows the KB is true**



## Limitation of the truth table approach.

$$KB \models \alpha \ ?$$

**Problem with the truth table approach:**

- the truth table is **exponential** in the number of propositional symbols (we checked all assignments)
- KB is true only on a small subset interpretations

**How to make the process more efficient?**

## Inference rules approach.

$$KB \models \alpha \ ?$$

**Problem with the truth table approach:**

- the truth table is **exponential** in the number of propositional symbols (we checked all assignments)
- KB is true on only a smaller subset

**How to make the process more efficient?**

**Solution: check only entries for which KB is *True*.**

This is the idea behind **the inference rules approach**

**Inference rules:**

- Represent sound inference patterns repeated in inferences
- Can be used to generate new (sound) sentences from the existing ones



## Inference rules for logic

- **Modus ponens**

$$\frac{A \Rightarrow B, \quad A}{B}$$

← **premise** (the line above the bar)

← **conclusion** (the line below the bar)

- If both sentences in the premise are true then conclusion is true.
- The modus ponens inference rule is **sound.**
  - We can prove this through the truth table.

| $A$ | $B$ | $A \Rightarrow B$ |
|---|---|---|
| *False* | *False* | *True* |
| *False* | *True* | *True* |
| *True* | *False* | *False* |
| *True* | *True* | *True* |

# Inference rules for logic

- **And-elimination**

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

- **And-introduction**

$$\frac{A_1, A_2, \quad A_n}{A_1 \wedge A_2 \wedge \quad A_n}$$

- **Or-introduction**

$$\frac{A_i}{A_1 \vee A_2 \vee \ldots A_i \vee \quad A_n}$$



# Inference rules for logic

- **Elimination of double negation**

$$\frac{\neg\neg A}{A}$$

- **Unit resolution**

$$\frac{A \vee B, \quad \neg A}{B}$$

A special case of

- **Resolution**

$$\frac{A \vee B, \quad \neg B \vee C}{A \vee C}$$

- All of the above inference rules **are sound.** We can prove this through the truth table, similarly to the **modus ponens** case.

## Example. Inference rules approach.

**KB:** $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** $S$

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$



## Example. Inference rules approach.

**KB:** $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** $S$

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. $P$ **From 1 and And-elim**

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

## Example. Inference rules approach.

**KB:** $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** $S$

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. $P$
5. $R$

**From 2,4 and Modus ponens**

$$\frac{A \Rightarrow B, \quad A}{B}$$



## Example. Inference rules approach.

**KB:** $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** $S$

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. $P$
5. $R$
6. $Q$

**From 1 and And-elim**

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

## Example. Inference rules approach.

**KB:** $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** $S$

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. $P$
5. $R$
6. $Q$
7. $(Q \wedge R)$ **From 5,6 and And-introduction**

$$\frac{A_1, A_2, \quad A_n}{A_1 \wedge A_2 \wedge \quad A_n}$$



## Example. Inference rules approach.

**KB:** $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** $S$

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. $P$
5. $R$
6. $Q$
7. $(Q \wedge R)$
8. $S$ **From 7,3 and Modus ponens**

$$\frac{A \Rightarrow B, \quad A}{B}$$

**Proved:** $S$

## Example. Inference rules approach.

**KB:** $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** $S$

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. $P$ — **From 1 and And-elim**
5. $R$ — **From 2,4 and Modus ponens**
6. $Q$ — **From 1 and And-elim**
7. $(Q \wedge R)$ — **From 5,6 and And-introduction**
8. $S$ — **From 7,3 and Modus ponens**

**Proved:** $S$

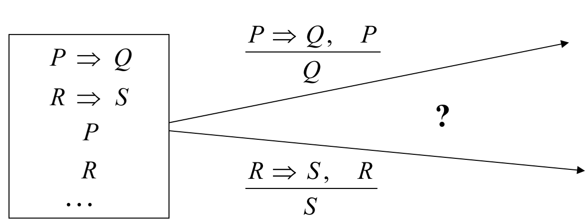


## Inference rules

- To show that theorem $\alpha$ holds for a KB
  - we may need to apply a number of sound inference rules

**Problem**: many possible inference rules to be applied next

**Looks familiar?**

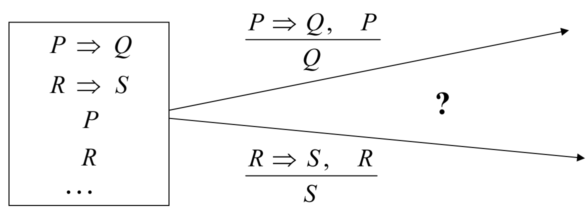


# Logic inferences and search

- To show that theorem $\alpha$ holds for a KB
  – we may need to apply a number of sound inference rules

**Problem**: many possible rules to can be applied next

**Looks familiar?**




**This is an instance of a search problem**:

**Truth table method (from the search perspective):**

– blind enumeration and checking



# Logic inferences and search

**Inference rule method as a search problem:**

- **State:** a set of sentences that are known to be true
- **Initial state**: a set of sentences in the KB
- **Operators**: applications of inference rules
  – Allow us to add new sound sentences to old ones
- **Goal state:** a theorem $\alpha$ is derived from KB

**Logic inference:**

- **Proof:** A sequence of sentences that are immediate consequences of applied inference rules
- **Theorem proving:** process of finding a proof of theorem