CS 1571 Introduction to AI
Lecture 13

First order logic

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Announcements

- **PS-5 due on Thursday**

- **Midterm exam:**
  – Thursday October 16, 2003
  – In class
  – Closed book
  – Covers Search and Logic
- **Office hours/recitations:**
  – Thursday 2:00-3:30pm
  – Friday 10:00-11:30am

First-order logic (FOL)

- More expressive than **propositional logic**
- **Eliminates deficiencies of PL by:**
  - Representing objects, their properties, relations and statements about them;
  - Introducing variables that refer to an arbitrary objects and can be substituted by a specific object
  - Introducing quantifiers allowing quantification statements over objects without the need to represent each of them separately

First-order logic. Syntax.

Term - syntactic entity for representing objects

Terms in FOL:

- **Constant symbols:**
  - E.g. *John, France, car89*
- **Variables:**
  - E.g. *x,y,z*
- **Functions** applied to one or more terms
  - E.g. *father-of (John)*
    father-of(father-of(John))

First order logic. Syntax.

Sentences in FOL:

- **Atomic sentences:**
  - **A predicate symbol** applied to 0 or more terms

    Examples:

    Red(car12),

    Sister(Amy, Jane);

    Manager(father-of(John));

  - t1 = t2 **equivalence** of terms

    Example:

    John = father-of(Peter)

First order logic. Syntax.

Sentences in FOL:

- **Complex sentences:**
- Assume ϕ, ψ are sentences. Then:
  - $(\phi \wedge \psi)$ $(\phi \vee \psi)$ $(\phi \Rightarrow \psi)$ $(\phi \Leftrightarrow \psi)$ $\neg \psi$

    and
  - $\forall x\ \phi$ $\exists y\ \phi$

  are sentences

Symbols $\exists, \forall$

- stand for the **existential** and the **universal** quantifier

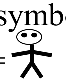
Semantics. Interpretation.

An interpretation *I* is defined by a **domain** and a **mapping**

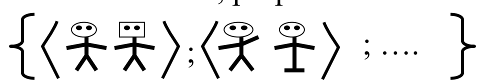
- **domain D:** a set of objects in the world we represent; domain of discourse;

An interpretation *I* maps:

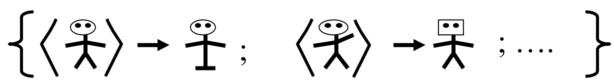
- Constant symbols to objects in D

  I(*John*) = 

- Predicate symbols to relations, properties on D

  I(*brother*) = 

- Function symbols to functional relations on D

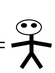
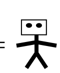
  I(*father-of*) = 

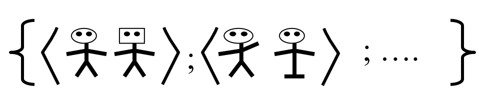
Semantics of sentences.

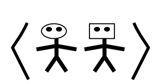
Meaning (evaluation) function:

V : sentence $\times$ interpretation $\rightarrow$ {*True* , *False* }

A **predicate** *predicate(term-1, term-2, term-3, term-n)* is true for the interpretation *I* , iff the objects referred to by *term-1, term-2, term-3, term-n* are in the relation referred to by *predicate*

I(*John*) =  I(*Paul*) = 

I(*brother*) = 

brother(John, Paul) =  in I(*brother*)

V(*brother(John, Paul), I*) = ***True***

Semantics of sentences.

- **Equality** V(*term-1*= *term-2, I*) = ***True***
  Iff I(*term-1*) =I(*term-2*)
- **Boolean expressions: standard**

E.g. V(*sentence-1* v *sentence-2, I*) = ***True***
Iff V(*sentence-1,I*)= ***True*** or V(*sentence-2,I*)= ***True***

- **Quantifications**

V($\forall x\ \phi$, *I*) = ***True***
Iff for all $d \in D$ V(ϕ ,$\underbrace{I[x/d]}_{\text{substitution of } x \text{ with } d}$)= ***True***

V($\exists x\ \phi$, *I*) = ***True***
Iff there is a $d \in D$, s.t. V(ϕ ,*I[x/d]*)= ***True***

Logical inference in FOL

Logical inference problem:

- Given a knowledge base KB (a set of sentences) and a sentence α , does the KB semantically entail α ?

$$KB \models \alpha \quad ?$$

In other words: In all interpretations in which sentences in the KB are true, is also α true?

Logical inference problem in the first-order logic is undecidable !!!. No procedure that can decide the entailment for all possible input sentences in a finite number of steps.

Truth table approach

- **Is it possible to modify the truth table approach also to the first-order logic (FOL)?**
- **Truth table approach:**
  - Generate all interpretations
  - Find the ones for which the KB evaluates to true
  - Check whether the theorem evaluates to true for all KB consistent interpretations
- **Not feasible !!!**

Inference rules approach

Advantage: Does not have to generate all possible interpretations

- **Inference rules from the propositional logic:**
  - **Modus ponens** $\dfrac{A \Rightarrow B, \quad A}{B}$
  - **Resolution** $\dfrac{A \vee B, \quad \neg B \vee C}{A \vee C}$
  - and others: And-introduction, And-elimination, Or-introduction, Negation elimination
- **Additional inference rules** are needed for sentences with quantifiers and variables
  - Must involve variable substitutions

Sentences with variables

First-order logic sentences can include variables.

- **Variable** is:
  - **Bound** – if it is in the scope of some quantifier
    $\forall x \; P(x)$
  - **Free** – if it is not bound.
    $\exists x \; P(y) \wedge Q(x)$ y is free
- **Sentence** (formula) is:
  - **Closed** – if it has no free variables
    $\forall y \exists x \; P(y) \Rightarrow Q(x)$
  - **Open** – if it is not closed
  - **Ground** – if it does not have any variables
    $Likes(John, Jane)$

Variable substitutions

- Variables in the sentences can be substituted with terms.
  (**terms = constants, variables, functions**)
- **Substitution:**
  - Is represented by a mapping from variables to terms
    $\{x_1 / t_1, x_2 / t_2, \ldots\}$
  - Application of the substitution to sentences

$$SUBST(\{x / Sam, y / Pam\}, Likes(x, y)) = Likes(Sam, Pam)$$

$$SUBST(\{x / z, y / fatherof(John)\}, Likes(x, y)) =$$
$$Likes(z, fatherof(John))$$

Inference rules for quantifiers

- **Universal elimination**

$$\frac{\forall x\ \phi(x)}{\phi(a)} \qquad a \text{ - is a constant symbol}$$

  – substitutes a variable with a constant symbol

$$\forall x\ Likes(x, IceCream) \qquad Likes(Ben, IceCream)$$

- **Existential elimination.**

$$\frac{\exists x\ \phi(x)}{\phi(a)}$$

  – Substitutes a variable with a constant symbol that does not appear elsewhere in the KB

$$\exists x\ Kill(x, Victim) \qquad Kill(Murderer, Victim)$$

Inference rules for quantifiers

- **Universal instantiation (introduction)**

$$\frac{\phi}{\forall x\ \phi} \qquad x - \text{is not free in } \phi$$

  – Introduces a universal variable which does not affect ϕ or its assumptions

$$Sister(Amy, Jane) \qquad \forall x\ Sister(Amy, Jane)$$

- **Existential instantiation (introduction)**

$$\frac{\phi(a)}{\exists x \phi(x)} \qquad \begin{array}{l} a - \text{is a ground term in } \phi \\ x - \text{is not free in } \phi \end{array}$$

  – Substitutes a ground term in the sentence with a variable and an existential statement

$$Likes(Ben, IceCream) \qquad \exists x\ Likes(x, IceCream)$$

Unification

- **Problem in inference:** Universal elimination gives many opportunities for substituting variables with ground terms

$$\frac{\forall x\, \phi(x)}{\phi(a)} \qquad a \text{ - is a constant symbol}$$

- **Solution:** Try substitutions that may help
  - Use substitutions of "similar" sentences in KB

- **Unification** – takes two similar sentences and computes the substitution that makes them look the same, if it exists

$$UNIFY\ (p,q) = \sigma \ \text{ s.t. } SUBST(\sigma, p) = SUBST\ (\sigma, q)$$

Unification. Examples.

- **Unification:**

$$UNIFY\ (p,q) = \sigma \ \text{ s.t. } SUBST(\sigma, p) = SUBST\ (\sigma, q)$$

- **Examples:**

$$UNIFY(Knows(John, x), Knows(John, Jane)) = \{x / Jane\}$$

$$UNIFY(Knows(John, x), Knows(y, Ann)) = ?$$

Unification. Examples.

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  $UNIFY(p,q) = \sigma$ s.t. $SUBST(\sigma, p) = SUBST(\sigma, q)$

- **Examples:**

$$UNIFY(Knows(John, x), Knows(John, Jane)) = \{x / Jane\}$$

$$UNIFY(Knows(John, x), Knows(y, Ann)) = \{x / Ann, y / John\}$$

$$UNIFY(Knows(John, x), Knows(y, MotherOf(y))) \\ = ?$$

Unification. Examples.

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$$UNIFY(Knows(John, x), Knows(y, Ann)) = \{x / Ann, y / John\}$$

$$UNIFY(Knows(John, x), Knows(y, MotherOf(y))) \\ = \{x / MotherOf(John), y / John\}$$

$$UNIFY(Knows(John, x), Knows(x, Elizabeth)) = ?$$

Unification. Examples.

- **Unification:**

  $UNIFY\ (p,q) = \sigma$ s.t. $SUBST(\sigma, p) = SUBST\ (\sigma, q)$

- **Examples:**

$UNIFY\ (Knows\ (John, x), Knows\ (John, Jane)) = \{x / Jane\}$

$UNIFY(Knows(John, x), Knows(y, Ann)) = \{x / Ann, y / John\}$

$$UNIFY\ (Knows\ (John, x), Knows\ (y, MotherOf\ (y))) \\ = \{x / MotherOf\ (John), y / John\}$$

$UNIFY\ (Knows\ (John, x), Knows\ (x, Elizabeth\ )) = fail$

Resolution inference rule

- **Recall:** Resolution inference rule is sound and complete (refutation-complete) for the **propositional logic** and CNF

$$\frac{A \vee B, \quad \neg A \vee C}{B \vee C}$$

- **Generalized resolution rule is sound and complete** (refutation-complete) for the first-order logic and CNF **w/o equalities**

$$\sigma = UNIFY\ (\phi_i, \neg \psi_j) \neq fail$$

$$\frac{\phi_1 \vee \phi_2 \ldots \vee \phi_k, \quad \psi_1 \vee \psi_2 \vee \ldots \psi_n}{SUBST(\sigma, \phi_1 \vee \ldots \vee \phi_{i-1} \vee \phi_{i+1} \ldots \vee \phi_k \vee \psi_1 \vee \ldots \vee \psi_{j-1} \vee \psi_{j+1} \ldots \psi_n)}$$

Example:
$$\frac{P(x) \vee Q(x), \quad \neg Q(John) \vee S(y)}{?}$$

Resolution inference rule

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Example: $$\frac{P(x) \vee Q(x), \quad \neg Q(John) \vee S(y)}{P(John) \vee S(y)}$$

Note: The rule is written in the **implicative form** in the book

Inference with the resolution rule

- **Proof by refutation:**
  - Prove that $KB, \neg \alpha$ is **unsatisfiable**
  - resolution is **refutation-complete**

- **Main procedure (steps):**
  1. Convert $KB, \neg \alpha$ to CNF with ground terms and universal variables only
  2. Apply repeatedly the resolution rule while keeping track and consistency of substitutions
  3. Stop when empty set (contradiction) is derived or no more new resolvents (conclusions) follow

Conversions to CNF

1. Eliminate implications, equivalences

$$(p \Rightarrow q) \rightarrow (\neg p \vee q)$$

2. Move negations inside (DeMorgan's Laws, double negation)

$$\neg(p \wedge q) \rightarrow \neg p \vee \neg q$$
$$\neg(p \vee q) \rightarrow \neg p \wedge \neg q$$
$$\neg \forall x \; p \rightarrow \exists x \; \neg p$$
$$\neg \exists x \; p \rightarrow \forall x \; \neg p$$
$$\neg\neg \; p \rightarrow p$$

3. Standardize variables (rename duplicate variables)

$$(\forall x \; P(x)) \vee (\exists x \; Q(x)) \rightarrow (\forall x \; P(x)) \vee (\exists y \; Q(y))$$

Conversion to CNF

4. Move all quantifiers left (no invalid capture possible)

$$(\forall x\ P(x)) \vee (\exists y\ Q(y)) \rightarrow \forall x\ \exists y\ P(x) \vee Q(y)$$

5. Skolemization (removal of existential quantifiers through elimination)

- If no universal quantifier occurs before the existential quantifier, replace the variable with a new constant symbol

$$\exists y\ P(A) \vee Q(y) \rightarrow P(A) \vee Q(B)$$

- If a universal quantifier precede the existential quantifier replace the variable with a function of the "universal" variable

$$\forall x\ \exists y\ P(x) \vee Q(y) \rightarrow \forall x\ \ P(x) \vee Q(F(x))$$

$F(x)$ - **a Skolem function**

Conversion to CNF

6. Drop universal quantifiers (all variables are universally quantified)

$$\forall x\ \ P(x) \vee Q(F(x)) \rightarrow P(x) \vee Q(F(x))$$

7. Convert to CNF using the distributive laws

$$p \vee (q \wedge r) \rightarrow (p \vee q) \wedge (p \vee r)$$

The result is a CNF with variables, constants, functions

Resolution example

$$\overbrace{\neg P(w) \vee Q(w), \neg Q(y) \vee S(y),\ P(x) \vee R(x), \neg R(z) \vee S(z)}^{\textbf{KB}},\quad \overset{\boldsymbol{\neg\alpha}}{\neg S(A)}$$

Resolution example

$$\overbrace{\neg P(w) \vee Q(w), \neg Q(y) \vee S(y),\ P(x) \vee R(x), \neg R(z) \vee S(z)}^{\textbf{KB}},\quad \overset{\boldsymbol{\neg\alpha}}{\neg S(A)}$$

$\{y / w\}$

$\neg P(w) \vee S(w)$

Resolution example

KB $\qquad\qquad$ $\neg\alpha$

$$\underbrace{\neg P(w) \vee Q(w), \neg Q(y) \vee S(y),\; P(x) \vee R(x), \neg R(z) \vee S(z)}_{}\,, \quad \neg S(A)$$

$\{y / w\}$

$$\neg P(w) \vee S(w)$$

$\{x / w\}$

$$S(w) \vee R(w)$$

Resolution example

KB $\qquad\qquad$ $\neg\alpha$

$$\neg P(w) \vee Q(w), \neg Q(y) \vee S(y),\; P(x) \vee R(x), \neg R(z) \vee S(z)\,, \quad \neg S(A)$$

$\{y / w\}$

$$\neg P(w) \vee S(w)$$

$\{x / w\}$

$$S(w) \vee R(w)$$

$\{z / w\}$

$$S(w)$$

Resolution example