CS 1571 Introduction to AI
Lecture 10

Propositional logic.

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Administration

- **Office hours for Tomas for September 26, 2003**
  - 2:00pm – 3:30pm
  - Only tomorrow, next week back on the regular schedule
- **PS-4:**
  - Material needed for Problem 2 and 3 is covered today

Knowledge-based agent

Knowledge base
Inference engine

- **Knowledge base (KB):**
  - A set of sentences that describe facts about the world in some formal (representational) language
  - **Domain specific**
- **Inference engine:**
  - A set of procedures that work upon the representational language and can infer new facts or answer KB queries
  - **Domain independent**

Knowledge representation

- The **objective of knowledge representation** is to express the knowledge about the world in a computer-tractable form

- Key aspects of knowledge representation languages:
  - **Syntax:** describes how sentences are formed in the language
  - **Semantics:** describes the meaning of sentences, what is it the sentence refers to in the real world
  - **Computational aspect:** describes how sentences and objects are manipulated in concordance with semantic conventions

Many KB systems rely on some variant of logic

Propositional logic. Syntax

Syntax:

- **Symbols (alphabet)** in P:
  - **Constants**: True, False
  - **A set of propositional variables** (propositional symbols):
    Examples: $P, Q, R, \ldots$ or statements like:
    Light in the room is on,
    It rains outside, etc.
  - **A set of logical connectives**:
    $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
- **Sentences**
  - **Build from symbols**

Propositional logic. Syntax

Sentences in the propositional logic:

- **Atomic sentences:**
  - **Constructed from constants and propositional symbols**
  - True, False are (atomic) sentences
  - P, Q or *Light in the room is on, It rains outside* are (atomic) sentences
- **Composite sentences:**
  - **Constructed from valid sentences via logical connectives**
  - If A, B are sentences then
    $\neg A \quad (A \wedge B) \quad (A \vee B) \quad (A \Rightarrow B) \quad (A \Leftrightarrow B)$
    or $(A \vee B) \wedge (A \vee \neg B)$
    are sentences

Semantic: interpretations

A **propositional symbol** (an atomic sentence) can stand for an arbitrary fact (statement) about the world

Examples: "*Light in the room is on*",

"*It rains outside*", etc.

An **interpretation:**

- maps symbols to one of the two values: ***True (T),*** or ***False (F)***,
- the value depends on the world we want to describe

World 1:

I: *Light in the room is on ->* ***True,*** *It rains outside ->* ***False***

World 2:

I': *Light in the room is on ->* ***False,*** *It rains outside ->* ***False***

Semantics: symbols and constants

- The **meaning (truth)** of the propositional symbol for a **specific interpretation** is given by its interpretation

V(Light in the room is on, **I**) = ***True***
V(Light in the room is on, **I'**) = ***False***

- **The meaning (truth) of constants:**
  - ***True*** and ***False*** constants are always (under any interpretation) assigned the corresponding ***True,False*** value

V(True, **I**) = ***True***
V(False, **I**) = ***False***

For any interpretation **I**

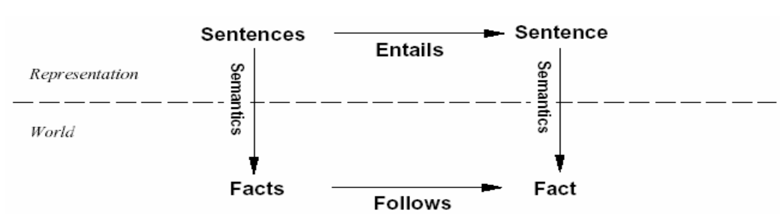
Semantics: composite sentences.

- **The meaning (truth value) of complex propositional sentences.**
  - Determined using the “standard” rules for combining logical sentences:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
True	*True*	*False*	*True*	*True*	*True*	*True*
True	*False*	*False*	*False*	*True*	*False*	*False*
False	*True*	*True*	*False*	*True*	*True*	*False*
False	*False*	*True*	*False*	*False*	*True*	*True*

Entailment

- **Entailment** reflects the relation of one fact in the world following from the others



- Entailment $KB \models \alpha$
- Knowledge base KB entails sentence α **if and only if** α is true in all worlds where KB is true

Inference.

- **Inference** is a process by which conclusions are reached.
- **Our goal:**

  We want to implement the inference process on a computer !!!

- Assume an **inference procedure *i*** that
  - derives a sentence α from the KB : $KB \vdash_i \alpha$

- **Important issue:**
  - **We need to assure that our inference procedure derives correct conclusions**

Sound and complete inference.

Assume an **inference procedure *i*** that

- derives a sentence α from the KB : $KB \vdash_i \alpha$

Properties of the inference procedure:

- **Soundness:** An inference procedure is **sound**

  If $KB \vdash_i \alpha$ then it is true that $KB \models \alpha$

- **Completeness:** An inference procedure is **complete**

  If $KB \models \alpha$ then it is true that $KB \vdash_i \alpha$

Logical inference problem

Logical inference problem:

- **Given:**
  - a knowledge base KB (a set of sentences) and
  - a sentence α (called **a theorem)**,
- **Does a KB semantically entail** α ? $KB \models \alpha$?

In other words: In all interpretations in which sentences in the KB are true, is also α true?

Question: Is there a procedure (program) that can decide this problem in a finite number of steps?

Answer: **Yes**. Logical inference problem for the propositional logic is **decidable**.

Solving logical inference problem

In the following:

How to design the procedure that answers:

$$KB \models \alpha \ ?$$

Three approaches:

- **Truth-table approach**
- **Inference rules**
- **Conversion to the inverse SAT problem**
  - **Resolution-refutation**

CS 1571 Intro to AI

Truth-table approach

Problem: $KB \models \alpha$?

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth tables:

- enumerate truth values of sentences for all possible interpretations (assignments of True/False values to propositional symbols)

Example:

		KB		α
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$
True	*True*	*True*	*True*	*True*
True	*False*	*True*	*False*	*False*
False	*True*	*True*	*False*	*False*
False	*False*	*False*	*True*	*False*

Truth-table approach

Problem: $KB \models \alpha$?

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth tables:

- enumerate truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

Example:

		KB		α
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$
True	*True*	*True*	*True*	*True*
True	*False*	*True*	*False*	*False*
False	*True*	*True*	*False*	*False*
False	*False*	*False*	*True*	*False*

Truth-table approach

Problem: $KB \models \alpha$?

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth tables:

- enumerate truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

Example:

		KB		α	
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$	
True	*True*	*True*	*True*	*True*	✓
True	*False*	*True*	*False*	*False*	
False	*True*	*True*	*False*	*False*	
False	*False*	*False*	*True*	*False*	

Truth-table approach

A two step procedure:

1. **Generate table for all possible interpretations**
2. Check whether the sentence α evaluates to true whenever KB evaluates to true

Example: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	*True*	*True*				
True	*True*	*False*				
True	*False*	*True*				
True	*False*	*False*				
False	*True*	*True*				
False	*True*	*False*				
False	*False*	*True*				
False	*False*	*False*				

Truth-table approach

A two steps procedure:

1. Generate table for all possible interpretations
2. **Check whether the sentence α evaluates to true whenever *KB* evaluates to true**

Example: $KB = (A \vee C) \wedge (B \vee \neg C) \qquad \alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	*True*	*True*	*True*	*True*	*True*	*True*
True	*True*	*False*	*True*	*True*	*True*	*True*
True	*False*	*True*	*True*	*False*	*False*	*True*
True	*False*	*False*	*True*	*True*	*True*	*True*
False	*True*	*True*	*True*	*True*	*True*	*True*
False	*True*	*False*	*False*	*True*	*False*	*True*
False	*False*	*True*	*True*	*False*	*False*	*False*
False	*False*	*False*	*False*	*True*	*False*	*False*

Truth-table approach

$KB = (A \vee C) \wedge (B \vee \neg C) \qquad \alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	*True*	*True*	*True*	*True*	*True*	*True*
True	*True*	*False*	*True*	*True*	*True*	*True*
True	*False*	*True*	*True*	*False*	*False*	*True*
True	*False*	*False*	*True*	*True*	*True*	*True*
False	*True*	*True*	*True*	*True*	*True*	*True*
False	*True*	*False*	*False*	*True*	*False*	*True*
False	*False*	*True*	*True*	*False*	*False*	*False*
False	*False*	*False*	*False*	*True*	*False*	*False*

KB entails α

- The **truth-table approach** is **sound and complete** for the propositional logic!!

Inference rules approach.

$$KB \models \alpha \;\; ?$$

Problem with the truth table approach:

- the truth table is **exponential** in the number of propositional symbols (we checked all assignments)
- KB is true on only a smaller subset

Idea: Can we check only entries for which KB is *True* ?

Solution: apply inference rules to sentences in the KB

Inference rules:

- Represent sound inference patterns repeated in inferences
- Can be used to generate new (sound) sentences from the existing ones

Inference rules for logic

- **Modus ponens**

$$\frac{A \Rightarrow B, \quad A}{B}$$

← **premise**

← **conclusion**

- If both sentences in the premise are true then conclusion is true.
- The modus ponens inference rule is **sound.**
  - We can prove this through the truth table.

A	B	$A \Rightarrow B$
False	*False*	*True*
False	*True*	*True*
True	*False*	*False*
True	*True*	*True*

Inference rules for logic

- **And-elimination**

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

- **And-introduction**

$$\frac{A_1, A_2, \quad A_n}{A_1 \wedge A_2 \wedge \quad A_n}$$

- **Or-introduction**

$$\frac{A_i}{A_1 \vee A_2 \vee \ldots A_i \vee \quad A_n}$$

Inference rules for logic

- **Elimination of double negation**

$$\frac{\neg\neg A}{A}$$

- **Unit resolution**

$$\frac{A \vee B, \quad \neg A}{B}$$

A special case of

- **Resolution**

$$\frac{A \vee B, \quad \neg B \vee C}{A \vee C}$$

- All of the above inference rules **are sound.** We can prove this through the truth table, similarly to the **modus ponens** case.

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P **From 1 and And-elim**

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R

From 2,4 and Modus ponens

$$\frac{A \Rightarrow B, \quad A}{B}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q

From 1 and And-elim

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q
7. $(Q \wedge R)$

From 5,6 and And-introduction

$$\frac{A_1, A_2, \quad A_n}{A_1 \wedge A_2 \wedge \quad A_n}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q
7. $(Q \wedge R)$
8. S

$$\frac{A \Rightarrow B, \quad A}{B}$$

From 7,3 and Modus ponens

Proved: S

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P — **From 1 and And-elim**
5. R — **From 2,4 and Modus ponens**
6. Q — **From 1 and And-elim**
7. $(Q \wedge R)$ — **From 5,6 and And-introduction**
8. S — **From 7,3 and Modus ponens**

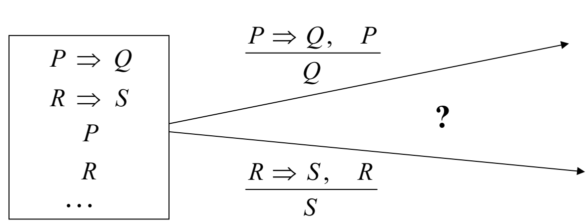
Proved: S

Inference rules

- To show that theorem α holds for a KB
  - we may need to apply a number of sound inference rules

Problem: many possible inference rules to be applied next

Does the problem look familiar?

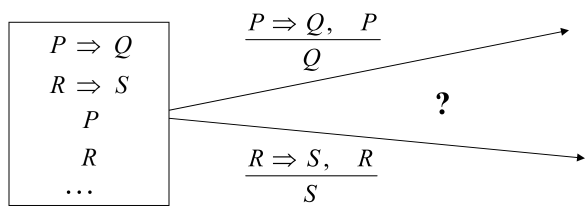


Logic inferences and search

- To show that theorem α holds for a KB
  - we may need to apply a number of sound inference rules

Problem: many possible inference rules to be applied next

Does the problem look familiar?



This is an instance of a search problem:

The truth-table method (from the search perspective):

- blind enumeration and checking

Logic inferences and search

Inference rule method as a search problem:

- **State:** a set of sentences that are known to be true
- **Initial state**: a set of sentences in the KB
- **Operators**: applications of inference rules
  - Allow us to add new sound sentences to old ones
- **Goal state:** a theorem α is derived from KB

Logic inference:

- **Proof:** A sequence of sentences that are immediate consequences of applied inference rules
- **Theorem proving:** process of finding a proof of theorem

Normal forms

Sentences in the propositional logic can be transformed into one of the normal forms. This can simplify the inferences.

Normal forms used:

Conjunctive normal form (CNF)

- conjunction of clauses (clauses include disjunctions of literals)

$$(A \vee B) \wedge (\neg A \vee \neg C \vee D)$$

Disjunctive normal form (DNF)

- Disjunction of terms (terms include conjunction of literals)

$$(A \wedge \neg B) \vee (\neg A \wedge C) \vee (C \wedge \neg D)$$

Conversion to a CNF

Assume: $\neg(A \Rightarrow B) \vee (C \Rightarrow A)$

1. Eliminate $\Rightarrow, \Leftrightarrow$

$$\neg(\neg A \vee B) \vee (\neg C \vee A)$$

2. Reduce the scope of signs through **DeMorgan Laws** and double negation

$$(A \wedge \neg B) \vee (\neg C \vee A)$$

3. Convert to the CNF using the associative and distributive laws

$$(A \vee \neg C \vee A) \wedge (\neg B \vee \neg C \vee A)$$

and

$$(A \vee \neg C) \wedge (\neg B \vee \neg C \vee A)$$

Satisfiability (SAT) problem

Determine whether a sentence in the conjunctive normal form (CNF) is satisfiable (i.e. can evaluate to true)

$$(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg R \vee S) \wedge (\neg P \vee Q \vee \neg T) \ldots$$

It is an instance of a constraint satisfaction problem:

- **Variables:**
  - Propositional symbols (*P, R, T, S*)
  - Values: ***True, False***
- **Constraints:**
  - Every conjunct must evaluate to true, at least one of the literals must evaluate to true
- **Why is this important?** All techniques developed for CSPs can be applied to solve the logical inference problem !!

Relationship between the inference problem and satisfiability

Inference problem:

- we want to show that the sentence α is entailed by KB

Satisfiability:

- The sentence is satisfiable if there is some assignment (interpretation) under which the sentence evaluates to true

Connection:

$KB \models \alpha$ if and only if $(KB \wedge \neg \alpha)$ is **unsatisfiable**

Consequences:

- inference problem is NP-complete
- programs for solving the SAT problem can be used to solve the inference problem