CS 1571 Introduction to AI
Lecture 8

Propositional logic

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

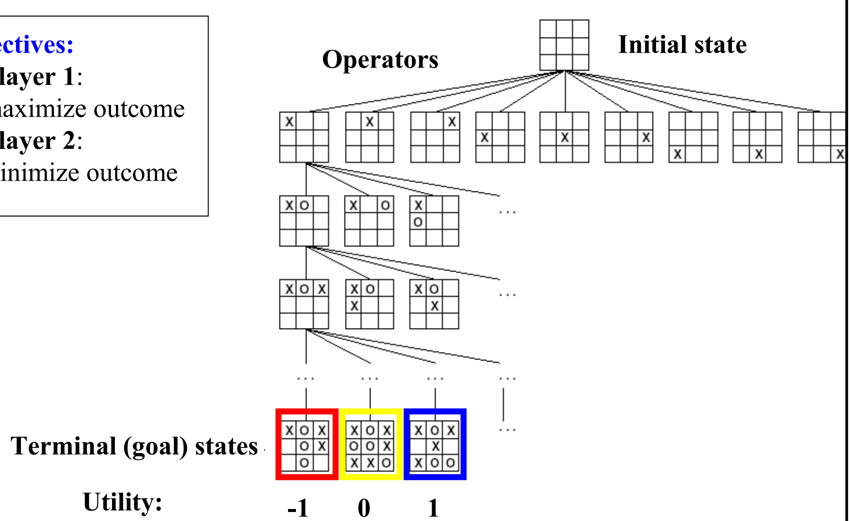
Game search

Game search problem

- **Game problem formulation:**
  - **Initial state:** initial board position + info whose move it is
  - **Operators:** legal moves a player can make
  - **Goal (terminal test):** determines when the game is over
  - **Utility (payoff) function**: measures the outcome of the game and its desirability
- **Search objective:**
  - find the sequence of player's decisions (moves) maximizing its utility (payoff)
  - Consider the opponent's moves and their utility

Game problem formulation (Tic-tac-toe)

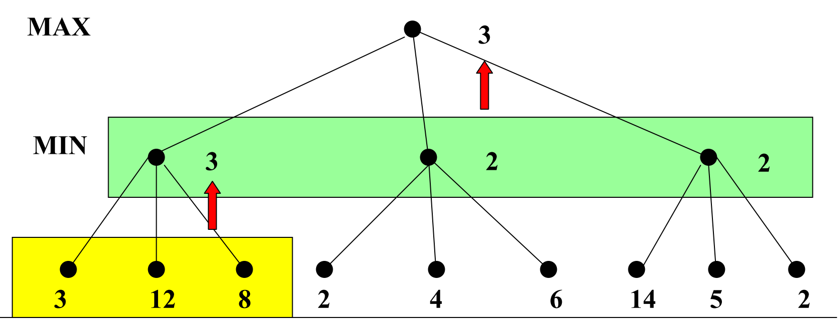
Objectives:
- **Player 1**: maximize outcome
- **Player 2**: minimize outcome



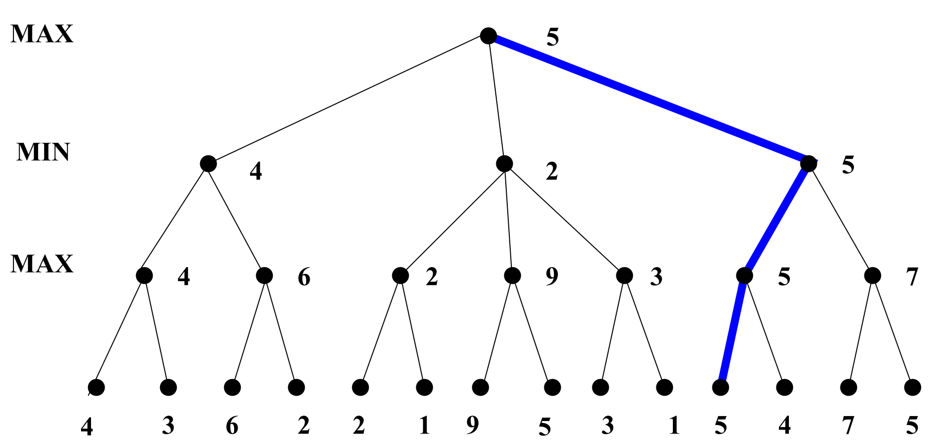
Minimax algorithm

How to deal with the contingency problem?

- Assuming that the opponent is rational and always optimizes its behavior (opposite to us) we consider the best opponent's response
- Then the **minimax algorithm** determines the best move

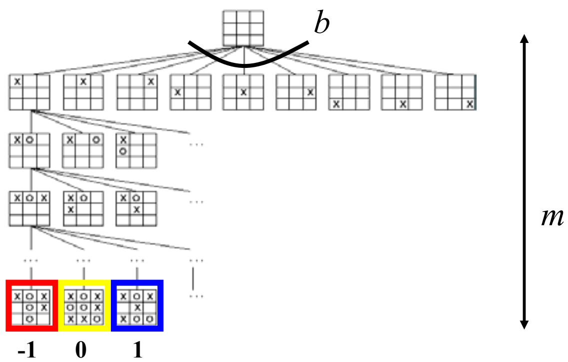


Minimax algorithm. Example



Complexity of the minimax algorithm

- We need to explore the complete game tree before making the decision



Complexity:

$O(b^m)$

- Impossible for large games
  - Chess: 35 operators, game can have 50 or more moves

Solution to the complexity problem

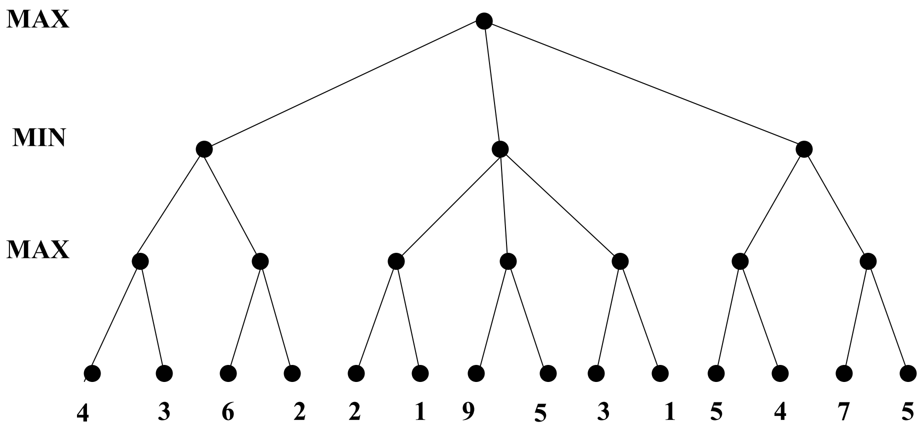
Two solutions:

1. **Dynamic pruning of redundant branches** of the search tree
   - identify provably suboptimal branch of the search tree even before it is fully explored
   - Cutoff the suboptimal branch

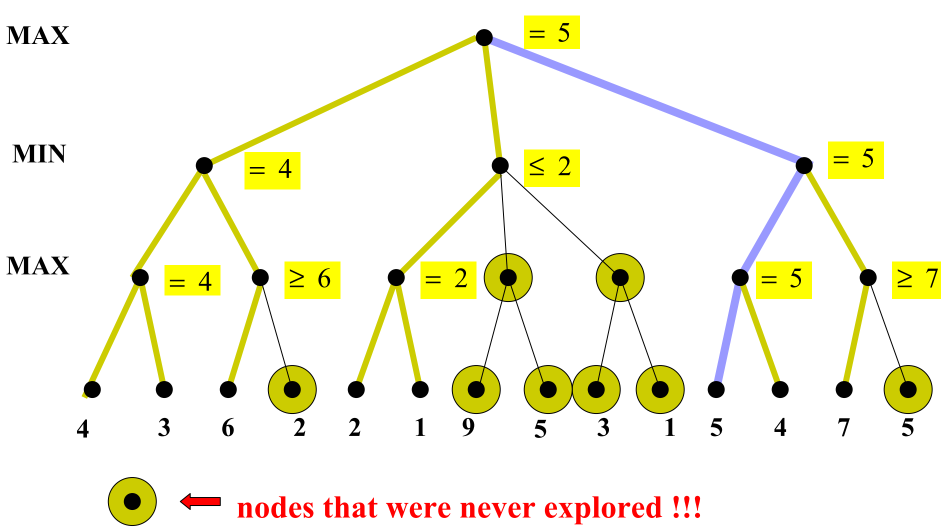
   Procedure: Alpha-Beta pruning

2. **Early cutoff of the search tree**
   - uses imperfect minimax value estimate of non-terminal states.

Alpha beta pruning. Example

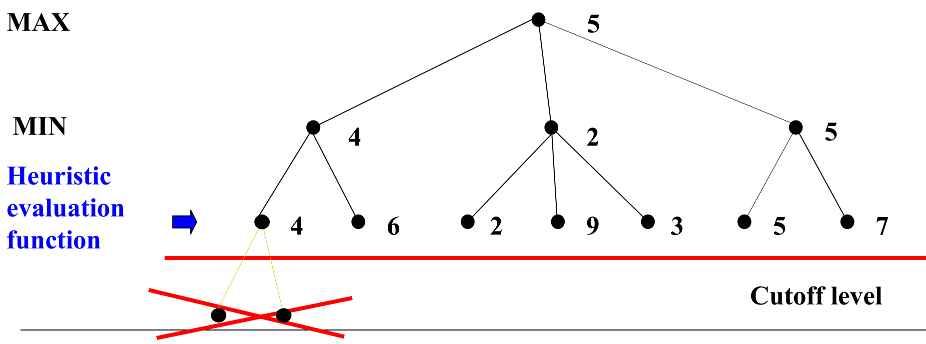


Alpha beta pruning. Example



Search tree cuttoffs

- **Idea:**
  - Cutoff the search tree before the terminal state is reached
  - Use imperfect estimate of the minimax value at leaves
    - Heuristic evaluation function
  - Select one move - repeat before every move



Heuristic evaluation functions

- Gives a **heuristic estimate** of the value for a sub-tree
- **Example of a heuristic functions**:
  - Material advantage in chess, checkers
    - Gives a value to every piece on the board, its position and combines them
  - More general **feature-based evaluation function**
    - Typically a linear evaluation function:

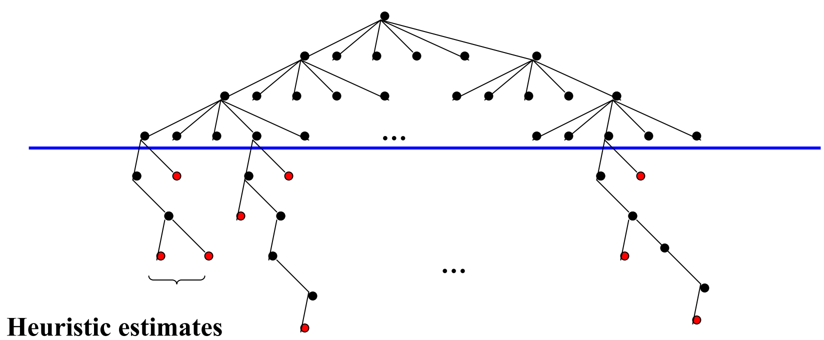
$$f(s) = f_1(s)w_1 + f_2(s)w_2 + \dots f_k(s)w_k$$

$f_i(s)$ - a feature of a state s

w_i - feature weight

Evaluation functions

- **Even better heuristic estimate** of the value for a sub-tree
- Restricted set of moves to be considered under **the cutoff level**
  - reduces branching and improves the evaluation function
  - Example: consider only the capture moves in chess



Knowledge representation:

Propositional logic

Knowledge-based agent

Knowledge base
Inference engine

- **Knowledge base (KB):**
  - A set of sentences that describe facts about the world in some formal (representational) language
  - Domain specific
- **Inference engine:**
  - A set of procedures that work upon the representational language and can infer new facts or answer KB queries
  - Domain independent

Example: MYCIN

- MYCIN: an expert system for diagnosis of bacterial infections
- **Knowledge base** represents
  - Facts about a specific patient case
  - Rules describing relations between entities in the bacterial infection domain

If	1. The stain of the organism is gram-positive, and
	2. The morphology of the organism is coccus, and
	3. The growth conformation of the organism is chains
Then	the identity of the organism is streptococcus

- **Inference engine:**
  - manipulates the facts and known relations to answer diagnostic queries (consistent with findings and rules)

Knowledge representation

- The objective of knowledge representation is to express the knowledge about the world in a computer-tractable form
- Key aspects of knowledge representation languages:
  - **Syntax:** describes how sentences are formed in the language
  - **Semantics:** describes the meaning of sentences, what is it the sentence refers to in the real world
  - **Computational aspect:** describes how sentences and objects are manipulated in concordance with semantical conventions

Many KB systems rely on some variant of logic

Logic

A formal language for expressing knowledge and ways of reasoning.

Logic is defined by**:**

- **A set of sentences**
  - A sentence is constructed from a set of primitives according to syntax rules.
- **A set of interpretations**
  - An interpretation gives a semantic to primitives. It associates primitives with values.
- **The valuation (meaning) function** V
  - Assigns a value (typically the truth value) to a given sentence under some interpretation

$$V : \text{sentence} \times \text{interpretation} \rightarrow \{True, False\}$$

Example of logic

Language of numerical constraints:

- **A sentence**:

  $x + 3 \leq z$

  x, z - variable symbols (primitives in the language)

- **An interpretation**:

  I: $x = 5, z = 2$

  Variables mapped to specific real numbers

- **Valuation (meaning) function** V:

  $V(x + 3 \leq z, \mathbf{I})$ is ***False*** for I: $x = 5, z = 2$

  is ***True*** for I: $x = 5, z = 10$

Types of logic

- Different types of logics possible:
  - Propositional logic
  - First-order logic
  - Temporal logic
  - Numerical constraints logic
  - Map-coloring logic

In the following:

- **Propositional logic.**
  - Formal language for making logical inferences
  - Foundations of **propositional logic**: **George Boole** (1854)

Propositional logic. Syntax

- **Propositional logic P:**
  - defines a language for symbolic reasoning

First step: define **Syntax+interpretation+semantics of P**

Syntax:

- **Symbols (alphabet)** in P:
  - **Constants**: True, False
  - **A set of propositional variables** (propositional symbols):
    Examples: $P, Q, R, \ldots$ or statements like:
    Light in the room is on,
    It rains outside, etc.
  - **A set of connectives**:
    $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

Propositional logic. Syntax

Sentences in the propositional logic:

- **Atomic sentences:**
  - **Constructed from constants and propositional symbols**
  - True, False are (atomic) sentences
  - P, Q or *Light in the room is on, It rains outside* are (atomic) sentences
- **Composite sentences:**
  - **Constructed from valid sentences via connectives**
  - If A, B are sentences then
    $\neg A \quad (A \wedge B) \quad (A \vee B) \quad (A \Rightarrow B) \quad (A \Leftrightarrow B)$
    or $(A \vee B) \wedge (A \vee \neg B)$
    are sentences

Propositional logic. Semantics.

The semantic gives the meaning to sentences.

In the propositional logic the semantics is defined by:

1. **Interpretation of propositional symbols and constants**
   – Semantics of atomic sentences
2. **Through the meaning of connectives**
   – Meaning (semantics) of composite sentences

Semantic: propositional symbols

A **propositional symbol** (an atomic sentence) can stand for an arbitrary fact (statement) about the world

Examples: "*Light in the room is on*",

"*It rains outside*", etc.

- An **interpretation** maps symbols to one of the two values: ***True (T),*** or ***False (F)***, depending on whether the symbol is satisfied in the world

  I: *Light in the room is on -> **True**, It rains outside -> **False***

  I': *Light in the room is on -> **False**, It rains outside -> **False***

- The **meaning (value)** of the propositional symbol for a specific interpretation is given by its interpretation

  V(Light in the room is on, **I**) = ***True***

  V(Light in the room is on, **I'**) = ***False***

Semantics: constants

- **The meaning (truth) of constants:**
  - True and False constants are always (under any interpretation) assigned the corresponding ***True,False*** value

$$\left.\begin{array}{l} V(True, \mathbf{I}) = \boldsymbol{True} \\ V(False, \mathbf{I}) = \boldsymbol{False} \end{array}\right\} \text{For any interpretation } \mathbf{I}$$

Semantics: composite sentences.

- **The meaning (truth value) of complex propositional sentences.**
  - Determined using the following rules for combining sentences:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
True	***True***	***False***	***True***	***True***	***True***	***True***
True	***False***	***False***	***False***	***True***	***False***	***False***
False	***True***	***True***	***False***	***True***	***True***	***False***
False	***False***	***True***	***False***	***False***	***True***	***True***

Contradiction and Tautology

Some composite sentences may always (under any interpretation) evaluate to a single truth value:

- **Contradiction** (always ***False***)

$$P \wedge \neg P$$

- **Tautology** (always ***True***)

$$P \vee \neg P$$

$$\left.\begin{array}{l} \neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q) \\ \neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q) \end{array}\right\} \text{DeMorgan's Laws}$$

Model, validity and satisfiability

- A **model (in logic):** An interpretation is a model for a set of sentences if it assigns true to each sentence in the set.
- A sentence is **satisfiable** if it has a model;
  - There is at least one interpretation under which the sentence can evaluate to True.
- A sentence is **valid** if it is ***True*** in all interpretations
  - i.e., if its negation is **not satisfiable** (leads to contradiction)

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	***True***	***True***	***False***	***True***
True	***False***	***True***	***True***	***True***
False	***True***	***True***	***False***	***True***
False	***False***	***False***	***False***	***True***

Model, validity and satisfiability

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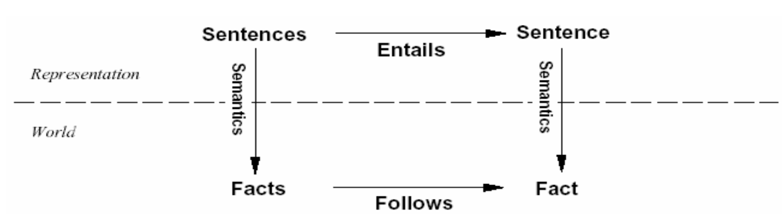
			Satisfiable sentence	Valid sentence
P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
False	*True*	*True*	*False*	*True*
False	*False*	*False*	*False*	*True*

Entailment

- The KB agent with reasoning capabilities should be able to generate new sentences (conclusions) that are true given the existing true sentences (stored in its knowledge base)
- **Entailment** reflects the relation of one fact in the world following from the others



- Entailment $KB \models \alpha$
- Knowledge base KB entails sentence α if and only if α is true in all worlds where KB is true

Sound and complete inference.

- **Inference** is a process by which conclusions are reached
- Assume an inference procedure I

  $KB \vdash_i \alpha$ = a sentence α can be derived from KB by i
- **Soundness:** An inference procedure is **sound**

  If $KB \vdash_i \alpha$ then it is true that $KB \models \alpha$
- **Completeness:** An inference procedure is **complete**

  If $KB \models \alpha$ then it is true that $KB \vdash_i \alpha$

Logical inference

Logical inference problem:

- Given a knowledge base KB (a set of sentences) and a sentence α, does a KB semantically entail α?

$$KB \models \alpha \quad ?$$

In other words: In all interpretations in which sentences in the KB are true, is also α true?

- Sentence α is also called **a theorem**

- **Logical inference problem for the propositional logic is decidable.**
  - There is a procedure that can answer the logical inference problem in a finite number of steps