CS 1571 Introduction to AI
Lecture 25

Logistic regression.

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Supervised learning

Data: $D = \{d_1, d_2, .., d_n\}$ **a set of *n* examples**

$d_i = < \mathbf{x}_i, y_i >$

$\mathbf{x}_i$ is input vector, and y is desired output (given by a teacher)

Objective: learn the mapping $f : X \rightarrow Y$

s.t. $y_i \approx f(x_i)$ for all $i = 1,.., n$

Two types of problems:

- **Regression:** Y is **continuous**
  Example: earnings, product orders → company stock price
- **Classification:** Y is **discrete**
  Example: temperature, heart rate → disease

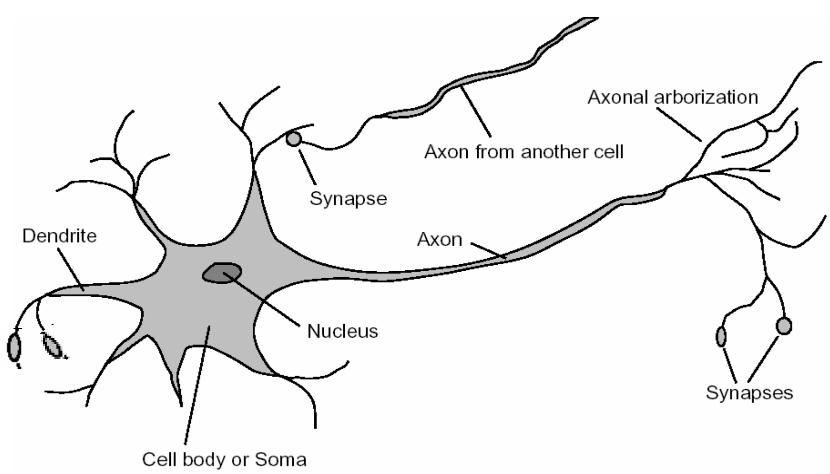
Today: **binary classification problems**

Binary classification

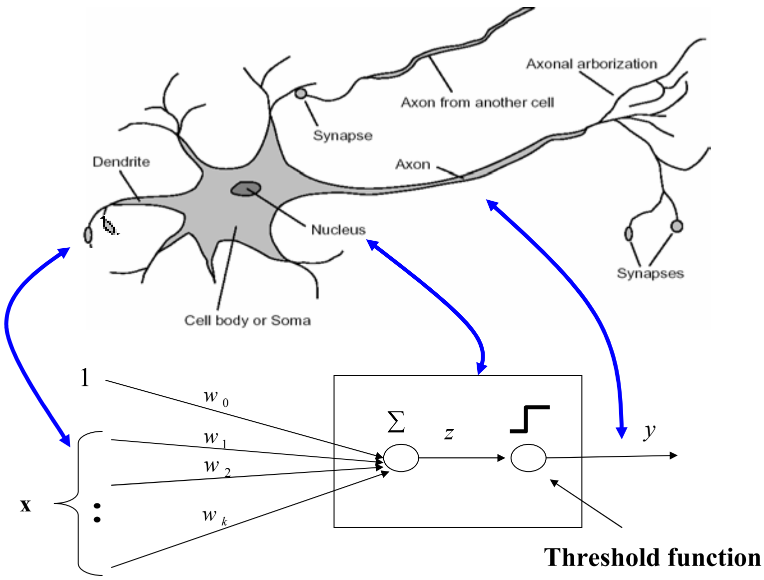
- **Two classes** $Y = \{0,1\}$
- Our goal is to learn how to classify correctly two types of examples
  - Class 0 – labeled as 0,
  - Class 1 – labeled as 1
- We would like to learn $f : X \to \{0,1\}$
- **First step:** we need to devise a model of the function f
- **Inspiration:** ***neuron (nerve cells)***

Neuron

- **neuron (nerve cell) and its activities**

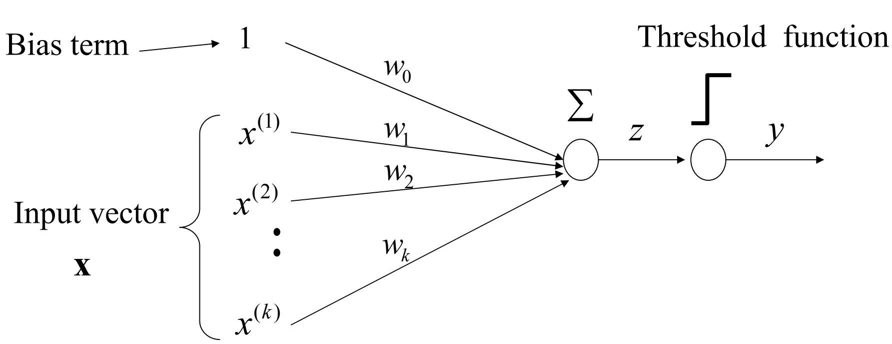


Neuron-based binary classification model



Neuron-based binary classification

- **Function we want to learn** $f : X \rightarrow \{0,1\}$



Binary classification

- Instead of learning the mapping to discrete values 0,1

$$f : X \rightarrow \{0,1\}$$

- It is easier to learn a probabilistic function

$$f' : X \rightarrow [0,1]$$

  - where *f'* describes the probability of a class 1 given x

$$p(y = 1 \mid \mathbf{x})$$

- Transformation back to discrete values:

If $p(y = 1 \mid \mathbf{x}) \geq 1/2$ then choose **1**
Else choose **0**

- **Logistic regression model** uses a probabilistic function

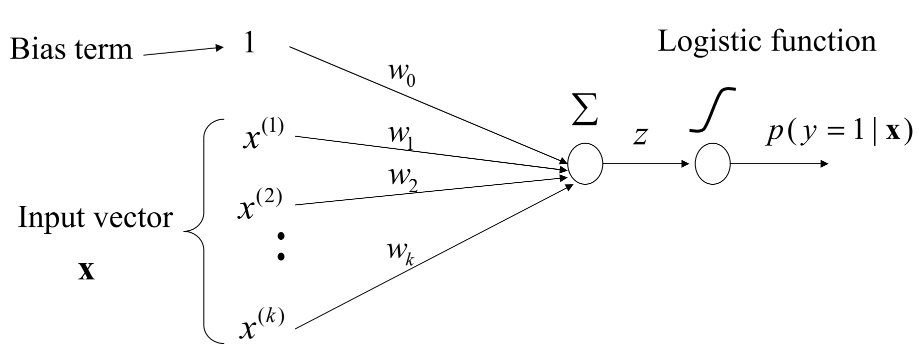
Logistic regression

- **Logistic regression:**

$$f(\mathbf{x}) = p(y = 1 \mid \mathbf{x}, \mathbf{w}) = g(w_0 + w_1 x^{(1)} + \ldots w_k x^{(k)})$$

where ***w*** are parameters of the models

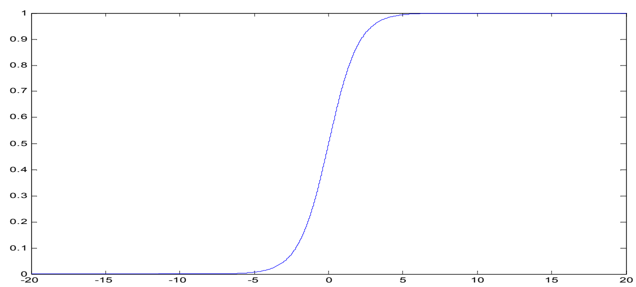
and *g(z)* is a **logistic function** $g(z) = 1/(1 + e^{-z})$



Logistic function

function $$g(z) = \frac{1}{(1+e^{-z})}$$

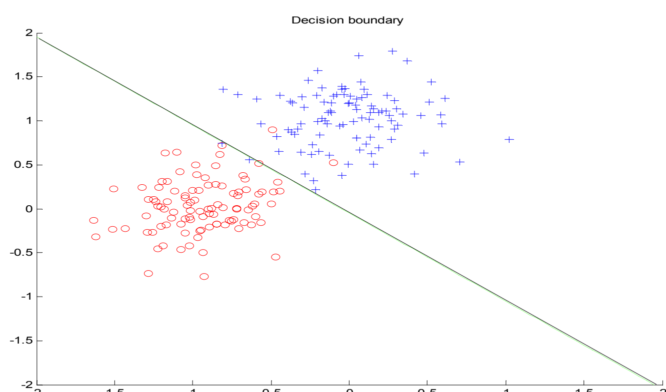
- also referred to as **sigmoid function**
- replaces threshold function with smooth switching
- takes a real number and outputs the number in the interval [0,1]



Logistic regression. Decision boundary

Logistic regression model defines a linear decision boundary

- **Example:** 2 classes (blue and red points)



Optimization of weights

- **Two classes:** $Y = \{0,1\}$
- **Data:** $D = \{d_1, d_2, .., d_n\}$

  $d_i =< \mathbf{x}_i, y_i >$
- We want to find the set of weight **w** that explain the data the best
  - weights that classify correctly as many examples as possible
- Zero-one error function

$$Error\ (x_i, y_i) = \begin{cases} 1 & f(\mathbf{x}_i, \mathbf{w}) \neq y_i \\ 0 & f(\mathbf{x}_i, \mathbf{w}) = y_i \end{cases}$$

- Error we would like to minimize: $E_{(x,y)}(Error\ (x, y))$
- The error is minimized if we choose:

$$y = 1 \quad \text{if} \quad p(y = 1 \mid \mathbf{x}, \mathbf{w}) > p(y = 0 \mid \mathbf{x}, \mathbf{w})$$
$$y = 0 \quad \text{otherwise}$$

Logistic regression. Parameter optimization.

- The error is minimized if we choose:

$$y = 1 \quad \text{if} \quad p(y = 1 \mid \mathbf{x}, \mathbf{w}) > p(y = 0 \mid \mathbf{x}, \mathbf{w})$$
$$y = 0 \quad \text{otherwise}$$

- We construct a probabilistic version of the error function based on the **likelihood of the data**

$$L(D, \mathbf{w}) = P(D \mid \mathbf{w})$$

- **Likelihood of the data**
  - Measures the goodness of fit

$$Error\ (D, \mathbf{w}) = -L(D, \mathbf{w})$$

Inverse optimization problem

Logistic regression: parameter learning.

- **Assume** $D_i =< \mathbf{x}_i, y_i >$
- **Let**

$$\mu_i = p(y_i = 1 \mid \mathbf{x}_i, \mathbf{w}) = g(z_i) = g(\mathbf{w}^T \mathbf{x})$$

- **Then**

$$L(D, \mathbf{w}) = \prod_{i=1}^{n} P(y = y_i \mid \mathbf{x}_i, \mathbf{w}) = \prod_{i=1}^{n} \mu_i^{y_i} (1 - \mu_i)^{1 - y_i}$$

- **Find weights w that maximize the likelihood of outputs**
  - log-likelihood trick The optimal weights are the same for both the likelihood and the log-likelihood

$$l(D, \mathbf{w}) = \log \prod_{i=1}^{n} \mu_i^{y_i} (1 - \mu_i)^{1 - y_i} = \sum_{i=1}^{n} \log \mu_i^{y_i} (1 - \mu_i)^{1 - y_i} =$$

$$= \sum_{i=1}^{n} y_i \log \mu_i + (1 - y_i) \log(1 - \mu_i) = \sum_{i=1}^{n} - J_{\text{online}}(D_i, \mathbf{w})$$

Logistic regression: parameter estimation

- **Log likelihood**

$$l(D, \mathbf{w}) = \sum_{i=1}^{n} - J_{\text{online}}(D_i, \mathbf{w}) = \sum_{i=1}^{n} y_i \log \mu_i + (1 - y_i) \log(1 - \mu_i)$$

- **On-line component of the log-likelihood**

$$- J_{\text{online}}(D_i, \mathbf{w}) = y_i \log \mu_i + (1 - y_i) \log(1 - \mu_i)$$

- **Derivatives of the online error component (in terms of weights)**

$$\frac{\partial}{\partial w_0} J_{\text{online}}(D_i, \mathbf{w}) = -(y_i - f(\mathbf{x}_i, \mathbf{w}))$$

$$\vdots$$

$$\frac{\partial}{\partial w_j} J_{\text{online}}(D_i, \mathbf{w}) = -(y_i - f(\mathbf{x}_i, \mathbf{w})) x_{i,j}$$

Logistic regression. Online gradient.

- We want to optimize the log-likelihood
- **On-line gradient update for the jth weight and ith step**

$$w_j^{(i)} \leftarrow w_j^{(i-1)} - \alpha \frac{\partial}{\partial w_j} [Error(D_i, \mathbf{w})|_{w^{(i-1)}}]$$

- **(i)th update for the logistic regression** $D_i = < \mathbf{x}_i, y_i >$

$$w_0^{(i1)} \leftarrow w_0^{(i-1)} + \alpha(i)(y_i - f(\mathbf{x}_i, \mathbf{w}^{(i-1)}))$$

$$\vdots$$

$$w_j^{(i)} \leftarrow w_j^{(i-1)} + \alpha(i)(y_i - f(\mathbf{x}_i, \mathbf{w}^{(i-1)}))x_{i,j}$$

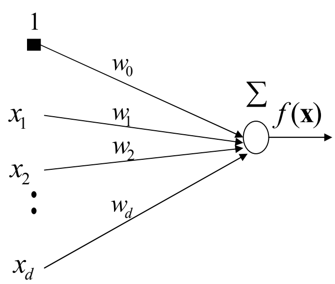
α - annealed learning rate (depends on the number of updates)

The same, easy update rule as used in the linear regression !!!

Online updates

Linear regression

$$f(\mathbf{x}) = \mathbf{w}^T \mathbf{x}$$

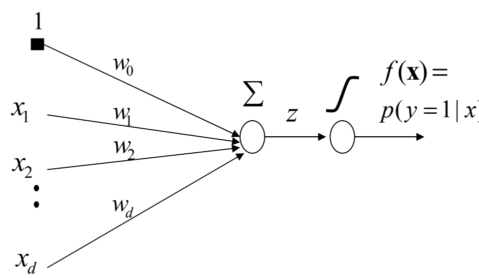


On-line gradient update:

$$\mathbf{w} \leftarrow \mathbf{w} + \alpha(y - f(\mathbf{x}, \mathbf{w}))\mathbf{x}$$

Logistic regression

$$f(\mathbf{x}) = p(y=1|\mathbf{x}, \mathbf{w}) = g(\mathbf{w}^T \mathbf{x})$$



On-line gradient update:

$$\mathbf{w} \leftarrow \mathbf{w} + \alpha(y - f(\mathbf{x}, \mathbf{w}))\mathbf{x}$$

The same

Online logistic regression algorithm

Online-logistic-regression (*D, number of iterations*)

initialize weights $w_0, w_1, w_2 \ldots w_k$

for i=1:1*: number of iterations*

do **select** a data point d=<$\mathbf{x}$,y> from D

set $\alpha = 1/i$

update weights (in parallel)

$$w_0 = w_0 + \alpha[y - f(\mathbf{x}, \mathbf{w})]$$

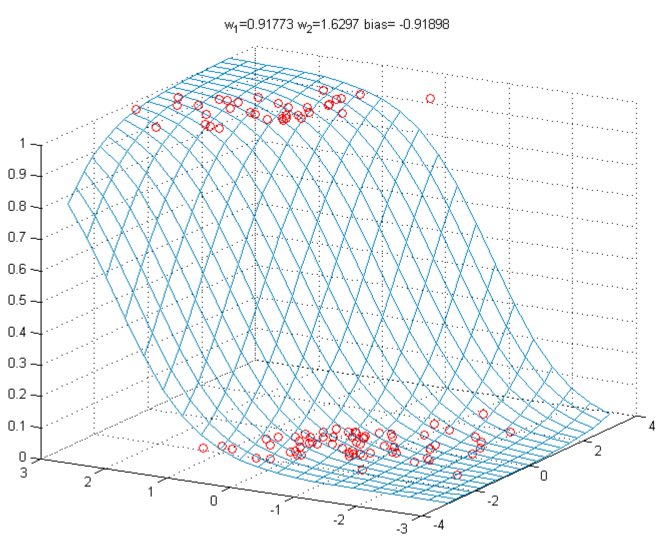
$$\vdots$$

$$w_j = w_j + \alpha[y - f(\mathbf{x}, \mathbf{w})]x_j$$

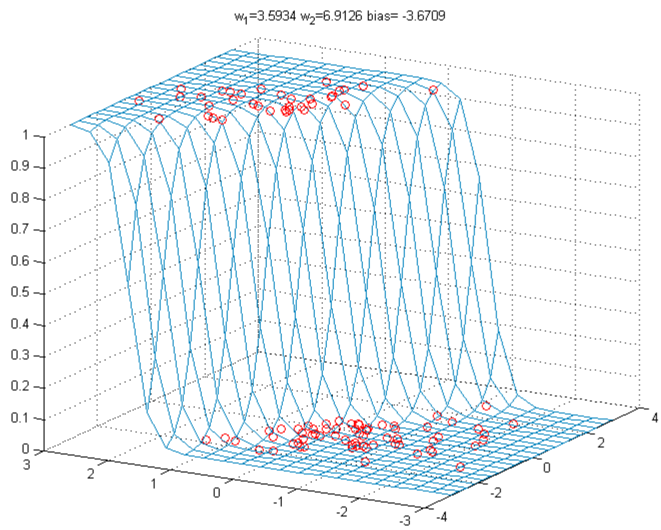
end for

return weights

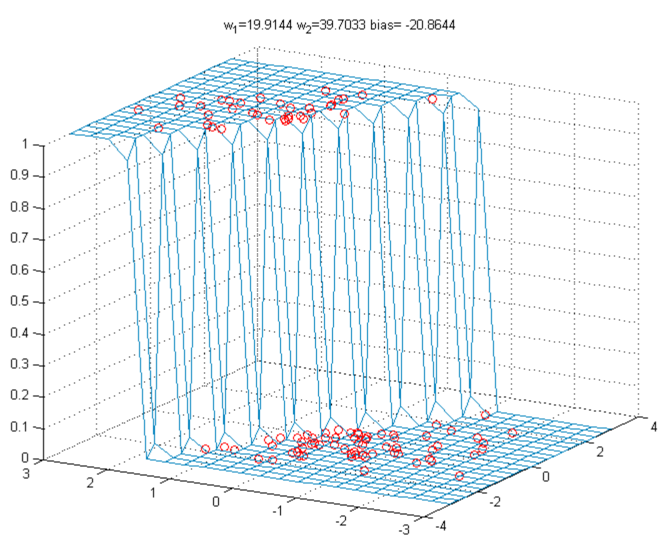
Online algorithm. Example.



Online algorithm. Example.



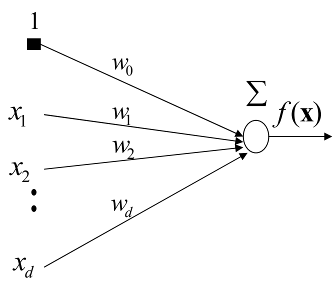
Online algorithm. Example.



Limitations of basic linear units

Linear regression

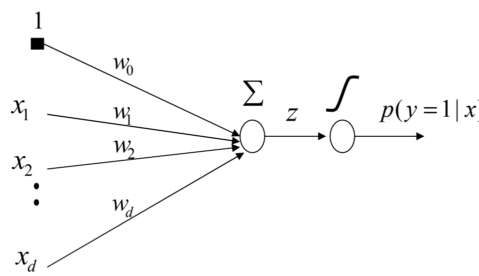
$$f(\mathbf{x}) = \mathbf{w}^T \mathbf{x}$$



Function linear in inputs !!

Logistic regression

$$f(\mathbf{x}) = p(y=1 \mid \mathbf{x}, \mathbf{w}) = g(\mathbf{w}^T \mathbf{x})$$

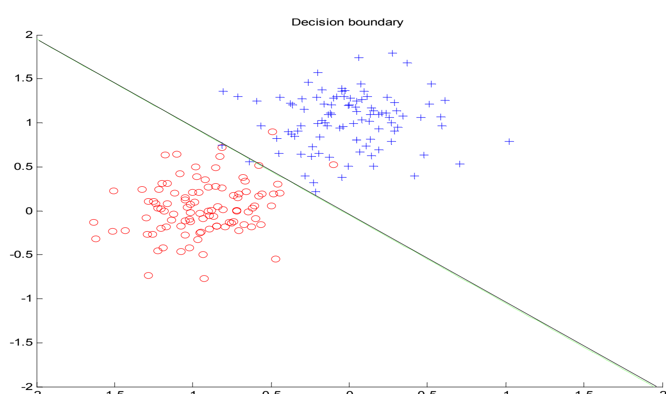


Linear decision boundary!!

Logistic regression. Decision boundary

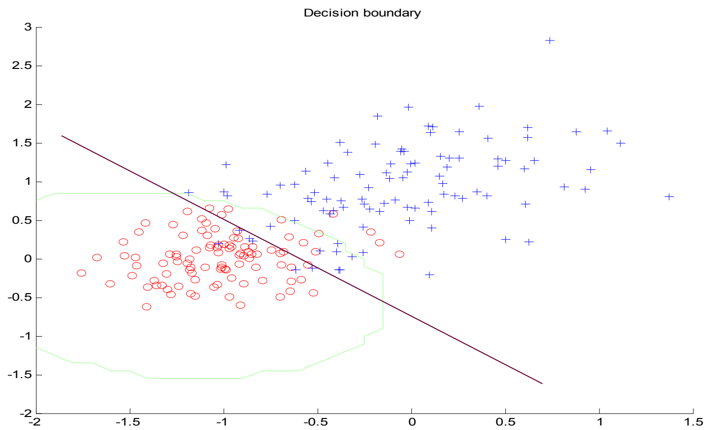
Logistic regression model defines a linear decision boundary

- Example: 2 classes (blue and red points)



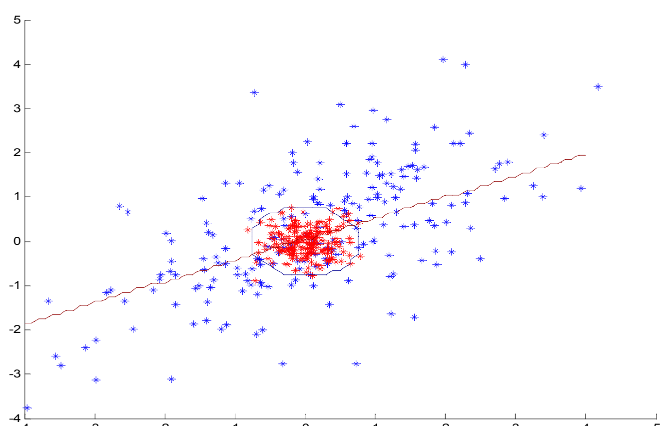
Linear decision boundary

- Example when logistic regression model is not optimal, but not that bad



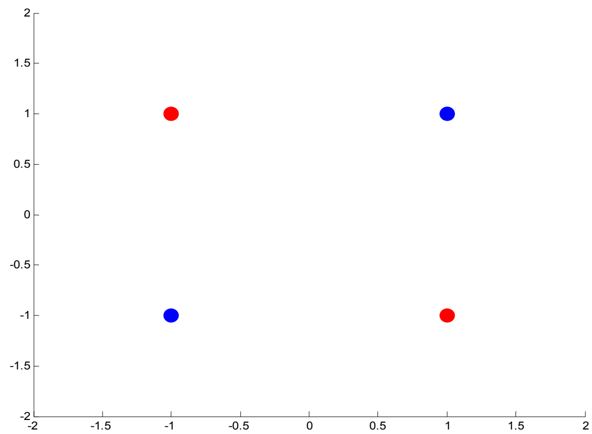
When logistic regression fails?

- Example in which the logistic regression model fails



Limitations of logistic regression.

- **parity function** - no linear decision boundary



Extensions of simple linear units

- use **feature (basis) functions** to model **nonlinearities**

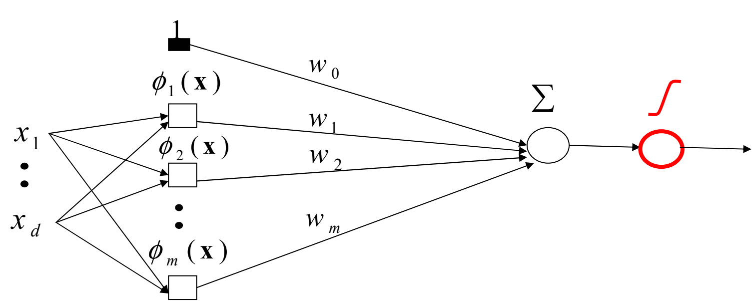
Linear regression

$$f(\mathbf{x}) = w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x})$$

Logistic regression

$$f(\mathbf{x}) = g(w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x}))$$

$\phi_j(\mathbf{x})$ - an arbitrary function of $\mathbf{x}$



The same trick can be done also for the logistic regression

Extension of simple linear units

- **Example:** **Fitting of a polynomial of degree m**
  - **Data points: pairs of** $< x, y >$
  - **Feature functions:**

$$\phi_i(x) = x^i$$

  - **Function to learn:**

$$f(x, \mathbf{w}) = w_0 + \sum_{i=1}^{m} w_i \phi_i(x) = w_0 + \sum_{i=1}^{m} w_i x^i$$

  - **On line update** for $<x,y>$ pair

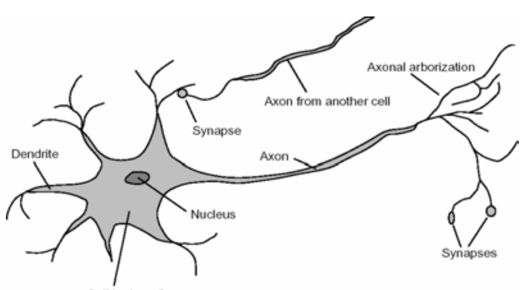
$$w_0 = w_0 + \alpha(y - f(\mathbf{x}, \mathbf{w}))$$

$$\vdots$$

$$w_j = w_j + \alpha(y - f(\mathbf{x}, \mathbf{w}))\phi_j(\mathbf{x})$$

Multi-layered neural networks

- Alternative way to introduce nonlinearities to regression/classification models
- **Idea:** Cascade several simple neural models (based on logistic regression). Much like neuron connections.



Next lecture !!!