CS 1571 Introduction to AI
Lecture 18

Bayesian belief networks

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Administration

- **Problem set 6 is due today**
- **Problem set 7 is out:**
  - **Due on November 5**
  - **No programming part**
- **Midterms:**
  - See the instructor
- **PS 1-5:**
  - See the TA

Modeling uncertainty with probabilities

- We need to define the full joint probability distribution over random variables defining the domain of interest
- With the known full joint we are able to handle an arbitrary probabilistic inference problem

Problems:

- **Space complexity.** To store a full joint distribution we need to remember $O(\mathrm{d}^n)$ numbers.

  n – number of random variables, d – number of values
- **Inference (time) complexity.** To compute some queries requires $O(\mathrm{d}^n)$ steps.
- **Acquisition problem.** Who is going to define all of the probability entries?

Medical diagnosis example.

- **Space complexity**.
  - Pneumonia (2 values: T,F), Fever (2: T,F), Cough (2: T,F), WBCcount (3: high, normal, low), paleness (2: T,F)
  - Number of assignments: 2*2*2*3*2=48
  - We need to define at least 47 probabilities.
- **Time complexity**.
  - Assume we need to compute the marginal of Pneumonia=T from the full joint distribution

$$P(Pneumonia = T) =$$

$$= \sum_{i \in T,F} \sum_{j \in T,F} \sum_{k=h,n,l} \sum_{u \in T,F} P(Pneumonia = T, Fever = i, Cough = j, WBCcount = k, Pale = u)$$

  - Sum over: 2*2*3*2=24 combinations

Bayesian belief networks (BBNs)

Bayesian belief networks.

- Represent the full joint distribution over the variables more compactly with a **smaller number of parameters**.
- Take advantage of **conditional and marginal independences** among random variables

- **A and B are independent**

$$P(A,B) = P(A)P(B)$$

- **A and B are conditionally independent given C**

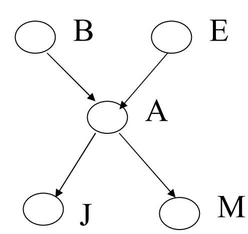
$$P(A,B \mid C) = P(A \mid C)P(B \mid C)$$

$$P(A \mid C,B) = P(A \mid C)$$

Bayesian belief networks (general)

Two components: $B = (S, \Theta_S)$

- **Directed acyclic graph**
  - Nodes correspond to random variables
  - (Missing) links encode independences



- **Parameters**
  - Local conditional probability distributions for every variable-parent configuration

$$\mathbf{P}(X_i \mid pa(X_i))$$

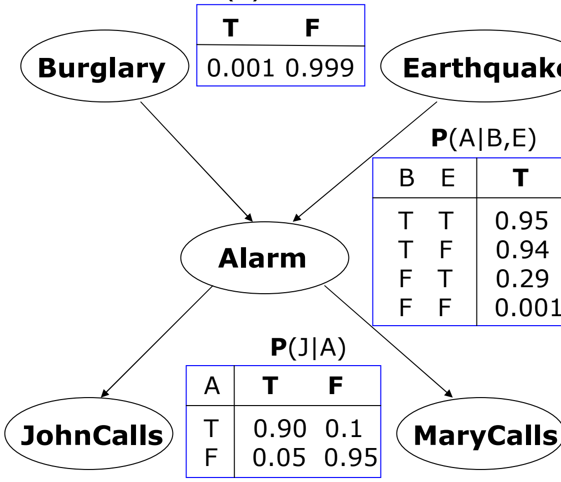
Where:

$pa(X_i)$ - stand for parents of X_i

P(A|B,E)

B	E	**T**	**F**
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

Bayesian belief network.



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

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P(J|A)

A	T	F
T	0.90	0.1
F	0.05	0.95

P(M|A)

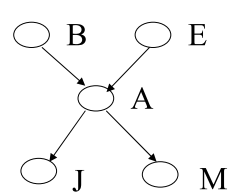
A	T	F
T	0.7	0.3
F	0.01	0.99

Full joint distribution in BBNs

Full joint distribution is defined in terms of local conditional distributions (obtained via the chain rule):

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

Example:



Assume the following assignment of values to random variables

$B = T, E = T, A = T, J = T, M = F$

Then its probability is:

$$P(B = T, E = T, A = T, J = T, M = F) =$$
$$P(B = T)P(E = T)P(A = T \mid B = T, E = T)P(J = T \mid A = T)P(M = F \mid A = T)$$

Bayesian belief networks (BBNs)

Bayesian belief networks

- Represents the full joint distribution over the variables more compactly using the product of local conditionals.
- **But how did we get to local parameterizations?**

Answer:

- **Graphical structure** encodes **conditional and marginal independences** among random variables
- **A and B are independent** $P(A,B) = P(A)P(B)$
- **A and B are conditionally independent given C**

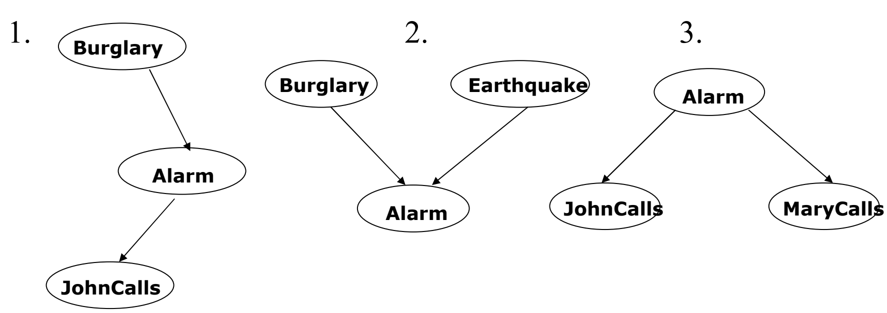
$$P(A \mid C,B) = P(A \mid C)$$

$$P(A,B \mid C) = P(A \mid C)P(B \mid C)$$

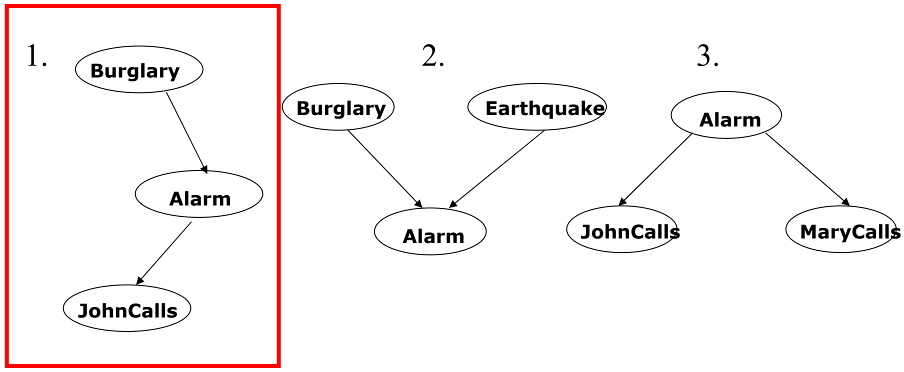
- **The graph structure implies the decomposition !!!**

Independences in BBNs

3 basic independence structures:



Independences in BBNs

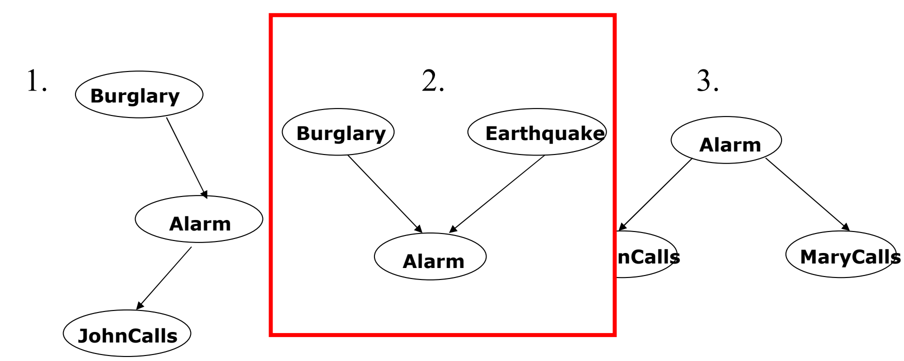


1. JohnCalls **is independent** of Burglary given Alarm

$$P(J \mid A, B) = P(J \mid A)$$

$$P(J, B \mid A) = P(J \mid A)P(B \mid A)$$

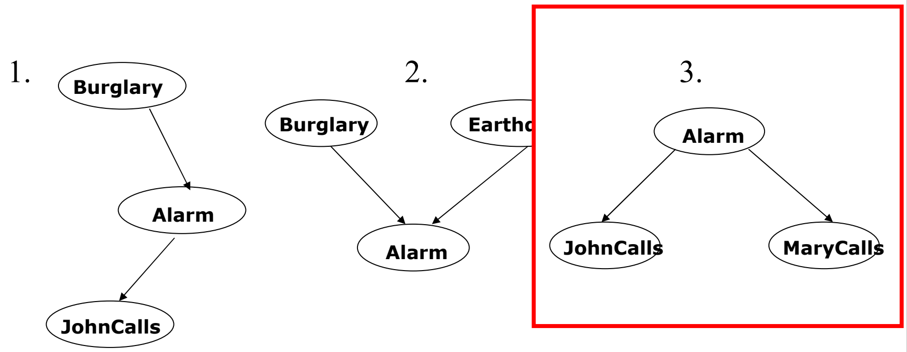
Independences in BBNs



2. Burglary **is independent** of Earthquake (not knowing about the Alarm) $P(B, E) = P(B)P(E)$

   But Burglary and Earthquake **become dependent** once I know the Alarm !!

Independences in BBNs



3. MaryCalls **is independent** of JohnCalls given Alarm

$$P(J \mid A, M) = P(J \mid A)$$

$$P(J, M \mid A) = P(J \mid A) P(M \mid A)$$

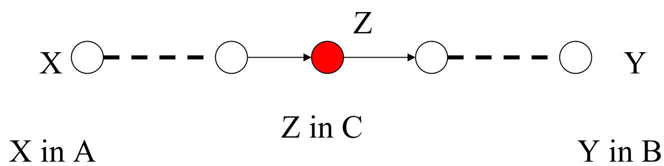
Independences in BBN

- BBN distribution models many conditional independence relations among distant variables and sets of variables
- These are defined in terms of the graphical criterion called d-separation
- **D-separation and independence**
  – Let X,Y and Z be three sets of nodes
  – If X and Y are d-separated by Z, then X and Y are conditionally independent given Z
- **D-separation :**
  – A is d-separated from B given C if every undirected path between them is **blocked**
- **Path blocking**
  – 3 cases that expand on the three basic independence structures

Undirected path blocking

A is d-separated from B given C if every undirected path between them is **blocked**

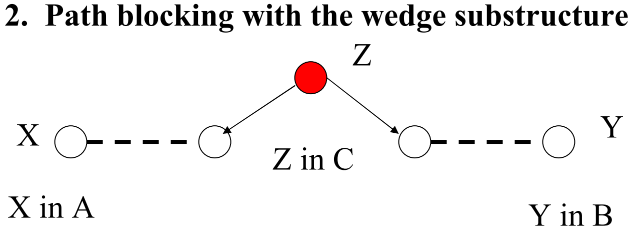
- **1. Path blocking with a linear substructure**



Undirected path blocking

A is d-separated from B given C if every undirected path between them is **blocked**

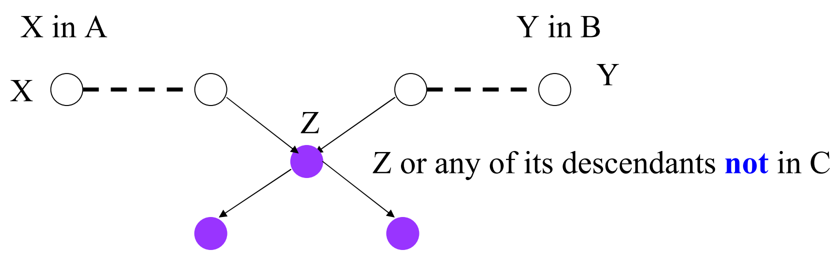
- **2. Path blocking with the wedge substructure**



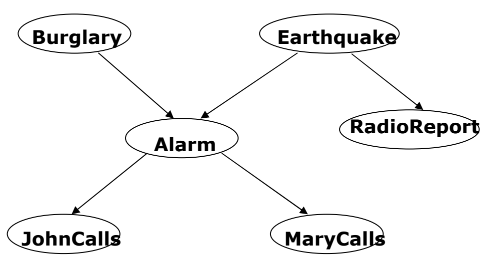
Undirected path blocking

A is d-separated from B given C if every undirected path between them is **blocked**

- **3. Path blocking with the vee substructure**

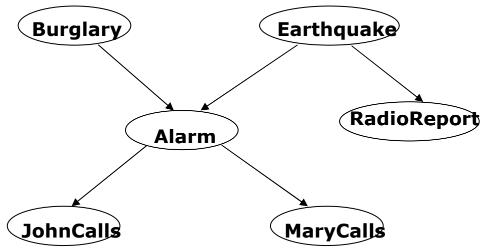


Independences in BBNs



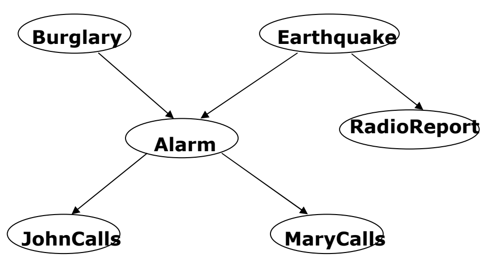
- Earthquake and Burglary are independent given MaryCalls ?

Independences in BBNs



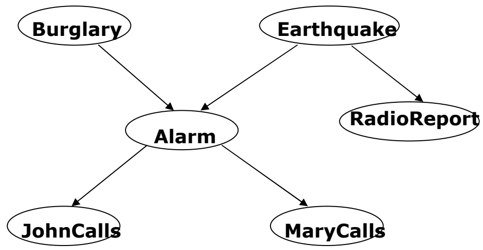
- Earthquake and Burglary are independent given MaryCalls **F**
- Burglary and MaryCalls are independent (not knowing Alarm) **?**

Independences in BBNs



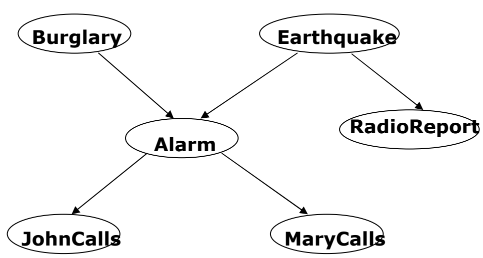
- Earthquake and Burglary are independent given MaryCalls **F**
- Burglary and MaryCalls are independent (not knowing Alarm) **F**
- Burglary and RadioReport are independent given Earthquake **?**

Independences in BBNs



- Earthquake and Burglary are independent given MaryCalls **F**
- Burglary and MaryCalls are independent (not knowing Alarm) **F**
- Burglary and RadioReport are independent given Earthquake **T**
- Burglary and RadioReport are independent given MaryCalls **?**

Independences in BBNs



- Earthquake and Burglary are independent given MaryCalls **F**
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- Burglary and RadioReport are independent given MaryCalls **F**

Bayesian belief networks (BBNs)

Bayesian belief networks

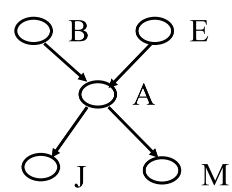
- Represents the full joint distribution over the variables more compactly using the product of local conditionals.
- **So how did we get to local parameterizations?**

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

- **The decomposition is implied by the set of independences encoded in the belief network.**

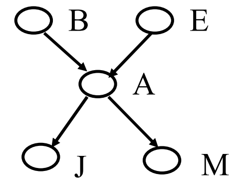
Full joint distribution in BBNs

Rewrite the full joint probability using the product rule:



$$P(B=T, E=T, A=T, J=T, M=F) =$$

Full joint distribution in BBNs

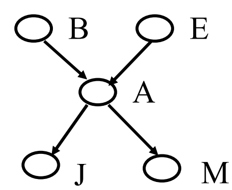


Rewrite the full joint probability using the product rule:

$$P(B=T,E=T,A=T,J=T,M=F)=$$

$$=P(J=T \mid B=T,E=T,A=T,M=F)P(B=T,E=T,A=T,M=F)$$
$$=\underline{P(J=T \mid A=T)}\underline{P(B=T,E=T,A=T,M=F)}$$

Full joint distribution in BBNs



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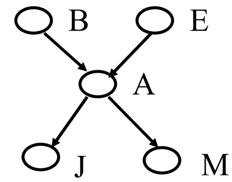
$$=\underline{P(J=T \mid A=T)}\underline{P(B=T,E=T,A=T,M=F)}$$
$$P(M=F \mid B=T,E=T,A=T)P(B=T,E=T,A=T)$$
$$\underline{P(M=F \mid A=T)}\underline{P(B=T,E=T,A=T)}$$

Full joint distribution in BBNs



Rewrite the full joint probability using the product rule:

$$P(B=T,E=T,A=T,J=T,M=F)=$$

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$$P(M=F\,|\,B=T,E=T,A=T)P(B=T,E=T,A=T)$$

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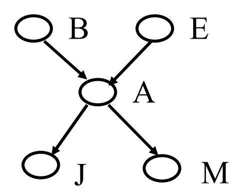
$$P(A=T\,|\,B=T,E=T)P(B=T,E=T)$$

Full joint distribution in BBNs



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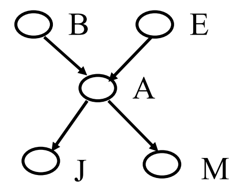
$$P(A=T\,|\,B=T,E=T)P(B=T,E=T)$$

$$P(B=T)P(E=T)$$

Full joint distribution in BBNs



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$$=P(J=T\mid A=T)P(B=T,E=T,A=T,M=F)$$

$$P(M=F\mid B=T,E=T,A=T)P(B=T,E=T,A=T)$$

$$P(M=F\mid A=T)P(B=T,E=T,A=T)$$

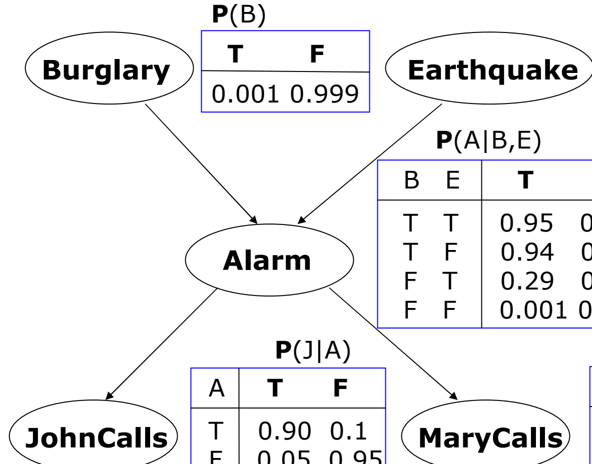
$$P(A=T\mid B=T,E=T)P(B=T,E=T)$$

$$P(B=T)P(E=T)$$

$$=P(J=T\mid A=T)P(M=F\mid A=T)P(A=T\mid B=T,E=T)P(B=T)P(E=T)$$

Bayesian belief network.

- In the BBN the **full joint distribution** is expressed using a set of local conditional distributions



P(B)

T	**F**
0.001	0.999

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P(A|B,E)

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P(M|A)

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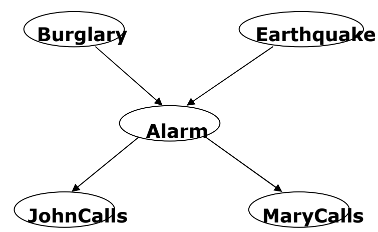
Parameter complexity problem

- In the BBN the **full joint distribution** is

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

- **What did we save?**



Parameters:
full joint: $2^5 = 32$

BBN: $2^3 + 2(2^2) + 2(2) = 20$

Parameters to be defined:
full joint: $2^5 - 1 = 31$

BBN: $2^2 + 2(2) + 2(1) = 10$

Model acquisition problem

The structure of the BBN

- typically reflects causal relations
  (BBNs are also sometime referred to as **causal networks)**
- Causal structure is intuitive in many applications domain and it is relatively easy to define to the domain expert

Probability parameters of BBN

- are conditional distributions relating random variables and their parents
- Complexity is much smaller than the full joint
- It is much easier to obtain such probabilities from the expert or learn them automatically from data