CS 1571 Introduction to AI
Lecture 11

Inference in first-order logic

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Administration

- **PS-4 due today:**
  - **Report**
  - **Programs**
- **PS-5 out:**
  - **Report**
  - **No programming assignment**
- **Midterm:**
  - **Tuesday, October 15, 2002**
  - **In-class, closed book**
  - **Material covered till (including) Thursday, October 10**

First-order logic (FOL)

- More expressive than **propositional logic**
- **Eliminates its deficiencies by:**
  - Representing objects, their properties, relations and statements about them;
  - Introducing variables that refer to an arbitrary objects and can be substituted by a specific object
  - Introducing quantifiers allowing quantification statements over groups of objects without the need to represent each of them separately
- **Predicate logic:** first-order logic without quantifiers

First-order logic. Syntax.

Term – a syntactic entity for representing objects

Terms in FOL:

- **Constant symbols:**
  - E.g. *John, France, car89*
- **Variables:**
  - E.g. *x,y,z*
- **Functions** applied to one or more terms
  - E.g. *father-of (John)*
    father-of(father-of(John))

First order logic. Syntax.

Sentences in FOL:

- **Atomic sentence:**
  - A predicate symbol applied to 0 or more terms
    e.g Red*(car123), Sister(Amy, Jane);
  - t1 = t2 equivalence of terms
    e.g. *John = father-of(Peter)*
- **Complex sentence:**
- Assume ϕ, ψ are sentences. Then:
  - $(\phi \wedge \psi)$ $(\phi \vee \psi)$ $(\phi \Rightarrow \psi)$ $(\phi \Leftrightarrow \psi)$ $\neg \psi$
    and
  - $\forall x \; \phi$ $\quad \exists y \; \phi$

  are sentences

  $\exists, \forall$ - stand for the existential and the universal quantifier

Examples of sentences with quantifiers

- **Universal quantification**

  All Upitt students are smart

  $\forall x \; student(x) \wedge at(x, Upitt) \Rightarrow smart(x)$

The main connective for the universal quantifier is implication

- **Existential quantification**

  Someone at CMU is smart

  $\exists x \; at(x, CMU) \wedge smart(x)$

The main connective for the existential quantifier is conjunction

Order of quantifiers

- **Quantifiers of the same type can be reordered**

  For all x and y, if x is a parent of y then y is a child of x

  $$\forall x, \forall y \;\; parent\;(x, y) \Rightarrow child\;(y, \mathrm{x})$$

  $$\forall y, \forall x \;\; parent\;(x, y) \Rightarrow child\;(y, \mathrm{x})$$

- **Order of different quantifiers changes the meaning**

  Everybody loves somebody

  $$\forall x \exists y \; loves(x, y)$$

  There is someone who is loved by everyone

  $$\exists y \forall x \; loves(x, y)$$

Connections between quantifiers

- A universal quantifier in the sentence can be expressed using existential quantifier and negations

  Everyone likes ice cream $\forall x \; likes(x, IceCream)$

  There is no one who does not like ice cream

  $$\neg \exists x \; \neg likes(x, IceCream)$$

- An existential quantifier in the sentence can be expressed using universal quantifier and negations

  Someone likes ice cream $\exists x \; likes(x, IceCream)$

  Not everyone does not like ice cream

  $$\neg \forall x \; \neg likes(x, IceCream)$$

Representing knowledge in FOL

Example:

Kinship domain

- **Objects:** people
  $John, Mary, Jane, \ldots$
- **Properties:** gender
  $Male(x), Female(x)$
- **Relations:** parenthood, brotherhood, marriage
  $Parent(x,y), Brother(x,y), Spouse(x,y)$
- **Functions:** mother-of (one for each person x)
  $MotherOf(x)$

Kinship domain in FOL

Relations between predicates and functions: write down what we know about them; how relate to each other.

- Male and female are disjoint categories
  $\forall x\; Male(x) \Leftrightarrow \neg Female(x)$
- Parent and child relations are inverse
  $\forall x,y\; Parent(x,y) \Leftrightarrow Child(y,x)$
- A grandparent is a parent of parent
  $\forall g,c\; Grandparent(g,c) \Leftrightarrow \exists p\; Parent(g,p) \wedge Parent(p,c)$
- A sibling is another child of one's parents
  $\forall x,y\; Sibling(x,y) \Leftrightarrow (x \neq y) \wedge \exists p\; Parent(p,x) \wedge Parent(p,y)$
- And so on ….

Logical inference in FOL

Logical inference problem:

- Given a knowledge base KB (a set of sentences) and a sentence α, does the KB semantically entail α?

$$KB \models \alpha \quad ?$$

In other words: In all interpretations in which sentences in the KB are true, is also α true?

Logical inference problem in the first-order logic is undecidable !!!. No procedure that can decide the entailment for all possible input sentences in a finite number of steps.

Inference rules

- **Inference rules from the propositional logic**:
  - Modus ponens

$$\frac{A \Rightarrow B, \quad A}{B}$$

  - Resolution

$$\frac{A \vee B, \quad \neg B \vee C}{A \vee C}$$

  - and others: And-introduction, And-elimination, Or-introduction, Negation elimination
- **Additional inference rules** are needed for sentences with quantifiers and variables
  - Must involve variable substitutions

Sentences with variables

First-order logic sentences can include variables.

- **Variable** is:
  - **Bound** – if it is in the scope of some quantifier

    $\forall x \; P(x)$
  - **Free** – if it is not bound.

    $\exists x \; P(y) \wedge Q(x)$ $\quad$ y is free
- **Sentence** (formula) is:
  - **Closed** – if it has no free variables

    $\forall y \exists x \; P(y) \Rightarrow Q(x)$
  - **Open** – if it is not closed
  - **Ground** – if it does not have any variables

    $Likes(John, Jane)$

Variable substitutions

- Variables in the sentences can be substituted with terms. (terms = constants, variables, functions)
- **Substitution:**
  - Is represented by a mapping from variables to terms

    $\{x_1 / t_1, x_2 / t_2, \ldots\}$
  - Application of the substitution to sentences

$$SUBST(\{x / Sam, y / Pam\}, Likes(x, y)) = Likes(Sam, Pam)$$

$$SUBST(\{x / z, y / fatherof(John)\}, Likes(x, y)) =$$
$$Likes(z, fatherof(John))$$

Inference rules for quantifiers

- **Universal elimination**

$$\frac{\forall x\ \phi(x)}{\phi(a)} \qquad a \text{ - is a constant symbol}$$

  – substitutes a variable with a constant symbol

$$\forall x\ Likes(x, IceCream) \qquad Likes(Ben, IceCream)$$

- **Existential elimination.**

$$\frac{\exists x\ \phi(x)}{\phi(a)}$$

  – Substitutes a variable with a constant symbol that does not appear elsewhere in the KB

$$\exists x\ Kill(x, Victim) \qquad Kill(Murderer, Victim)$$

Inference rules for quantifiers

- **Universal instantiation (introduction)**

$$\frac{\phi}{\forall x\ \phi} \qquad x \text{ – is not free in } \phi$$

  – Introduces a universal variable which does not affect ϕ or its assumptions

$$Sister(Amy, Jane) \qquad \forall x\ Sister(Amy, Jane)$$

- **Existential instantiation (introduction)**

$$\frac{\phi(a)}{\exists x \phi(x)} \qquad \begin{array}{l} a \text{ – is a ground term in } \phi \\ x \text{ – is not free in } \phi \end{array}$$

  – Substitutes a ground term in the sentence with a variable and an existential statement

$$Likes(Ben, IceCream) \qquad \exists x\ Likes(x, IceCream)$$

Unification

- **Problem in inference:** Universal elimination gives many opportunities for substituting variables with ground terms

$$\frac{\forall x\ \phi(x)}{\phi(a)} \qquad a \text{ - is a constant symbol}$$

- **Solution:** Try substitutions that may help
  - Use substitutions of "similar" sentences in KB

- **Unification** – takes two similar sentences and computes the substitution that makes them look the same, if it exists

$$UNIFY\ (p,q) = \sigma \text{ s.t. } SUBST(\sigma, p) = SUBST\ (\sigma, q)$$

Unification. Examples.

- **Unification:**

$$UNIFY\ (p,q) = \sigma \text{ s.t. } SUBST(\sigma, p) = SUBST\ (\sigma, q)$$

- **Examples:**

$$UNIFY\,(Knows\,(John, x), Knows\,(John, Jane)) = \{x / Jane\}$$

$$UNIFY(Knows(John, x), Knows(y, Ann)) = ?$$

$$UNIFY\,(Knows\,(John, x), Knows\,(y, MotherOf\ (y)))$$
$$= ?$$

$$UNIFY\,(Knows\,(John, x), Knows\,(x, Elizabeth\,)) = ?$$

Unification. Examples.

- **Unification:**

  $UNIFY\,(p,q)=\sigma$ s.t. $SUBST(\sigma,p)=SUBST\,(\sigma,q)$

- **Examples:**

$$UNIFY\,(Knows\,(John,x),Knows\,(John,Jane))=\{x/Jane\}$$

$$UNIFY(Knows(John,x),Knows(y,Ann))=\{x/Ann,y/John\}$$

$$UNIFY\,(Knows\,(John,x),Knows\,(y,MotherOf\,(y)))$$
$$=\{x/MotherOf\,(John),y/John\}$$

$$UNIFY\,(Knows\,(John,x),Knows\,(x,Elizabeth\,))=fail$$

Resolution inference rule

- **Recall:** Resolution inference rule is sound and complete (refutation-complete) for the **propositional logic** and CNF

$$\frac{A\vee B,\quad \neg A\vee C}{B\vee C}$$

- **Generalized resolution rule is sound and complete** (refutation-complete) for the first-order logic and CNF w/o equalities

$$\sigma=UNIFY\;(\phi_i,\neg\psi_j)\neq fail$$

$$\frac{\phi_1\vee\phi_2\ldots\vee\phi_k,\quad \psi_1\vee\psi_2\vee\ldots\psi_n}{SUBST(\sigma,\phi_1\vee\ldots\vee\phi_{i-1}\vee\phi_{i+1}\ldots\vee\phi_k\vee\psi_1\vee\ldots\vee\psi_{j-1}\vee\psi_{j+1}\ldots\psi_n)}$$

Example: $\dfrac{P(x)\vee Q(x),\quad \neg Q(John)\vee S(y)}{P(John)\vee S(y)}$

The rule can be also written in the **implicative form** (book)

Inference with resolution rule

- **Proof by refutation:**
  - Prove that $KB, \neg \alpha$ is **unsatisfiable**
  - resolution is **refutation-complete**

- **Main procedure (steps):**
  1. Convert $KB, \neg \alpha$ to CNF with ground terms and universal variables only
  2. Apply repeatedly the resolution rule while keeping track and consistency of substitutions
  3. Stop when empty set (contradiction) is derived or no more new resolvents (conclusions) follow

Conversion to CNF

1. Eliminate implications, equivalences

$$(p \Rightarrow q) \rightarrow (\neg p \vee q)$$

2. Move negations inside (DeMorgan's Laws, double negation)

$$\neg(p \wedge q) \rightarrow \neg p \vee \neg q$$
$$\neg(p \vee q) \rightarrow \neg p \wedge \neg q$$
$$\neg \forall x \; p \rightarrow \exists x \; \neg p$$
$$\neg \exists x \; p \rightarrow \forall x \; \neg p$$
$$\neg\neg \; p \rightarrow p$$

3. Standardize variables (rename duplicate variables)

$$(\forall x \; P(x)) \vee (\exists x \; Q(x)) \rightarrow (\forall x \; P(x)) \vee (\exists y \; Q(y))$$

Conversion to CNF

4. Move all quantifiers left (no invalid capture possible)

$$(\forall x\ P(x)) \vee (\exists y\ Q(y)) \rightarrow \forall x\ \exists y\ P(x) \vee Q(y)$$

5. Skolemization (removal of existential quantifiers through elimination)

- If no universal quantifier occurs before the existential quantifier, replace the variable with a new constant symbol
  $$\exists y\ P(A) \vee Q(y) \rightarrow P(A) \vee Q(B)$$
- If a universal quantifier precede the existential quantifier replace the variable with a function of the "universal" variable

$$\forall x\ \exists y\ P(x) \vee Q(y) \rightarrow \forall x\ \ P(x) \vee Q(F(x))$$

$F(x)$ - **a Skolem function**

Conversion to CNF

6. Drop universal quantifiers (all variables are universally quantified)

$$\forall x\ \ P(x) \vee Q(F(x)) \rightarrow P(x) \vee Q(F(x))$$

7. Convert to CNF using the distributive laws

$$p \vee (q \wedge r) \rightarrow (p \vee q) \wedge (p \vee r)$$

The result is a CNF with variables, constants, functions

Resolution example

KB $\qquad\qquad\qquad$ $\neg\alpha$

$$\overbrace{\neg P(w) \vee Q(w), \neg Q(y) \vee S(y), P(x) \vee R(x), \neg R(z) \vee S(z)}^{KB}, \quad \neg S(A)$$

Resolution example

KB $\qquad\qquad\qquad$ $\neg\alpha$

$$\overbrace{\neg P(w) \vee Q(w), \neg Q(y) \vee S(y), P(x) \vee R(x), \neg R(z) \vee S(z)}^{KB}, \quad \neg S(A)$$

$\{y / w\}$

$\neg P(w) \vee S(w)$

Resolution example

KB $\quad\quad$ **¬α**

$$\neg P(w) \vee Q(w), \neg Q(y) \vee S(y), P(x) \vee R(x), \neg R(z) \vee S(z), \quad \neg S(A)$$

$\{y/w\}$

$\neg P(w) \vee S(w)$

$\{x/w\}$

$S(w) \vee R(w)$

Resolution example

KB $\quad\quad$ **¬α**

$$\neg P(w) \vee Q(w), \neg Q(y) \vee S(y), P(x) \vee R(x), \neg R(z) \vee S(z), \quad \neg S(A)$$

$\{y/w\}$

$\neg P(w) \vee S(w)$

$\{x/w\}$

$\{z/w\}$

$S(w) \vee R(w)$

$S(w)$

Resolution example