Combinatorics: counting # of outcomes

Lecture 5

Computing probabilities without enumerating all the outcomes

- To compute probability of the discrete event A:
  - Compute the total number of all possible outcomes: *N (All)*
  - Compute the number *T* of possible outcomes where event A is true

  - The probability P(A)= *T*/*N*

- To compute the number of outcomes we will use combinatorics

Counting rule #1: Rule of sum

- If we need to count the total number of elements in 2 sets: sum up set cardinalities
- Example:
  - P: set of pizzerias, $|P| = 5$
  - R: set of restaurants, $|R| = 7$
  - How many total places to eat?

Exercise 1: counting integers

Use rule of sum!

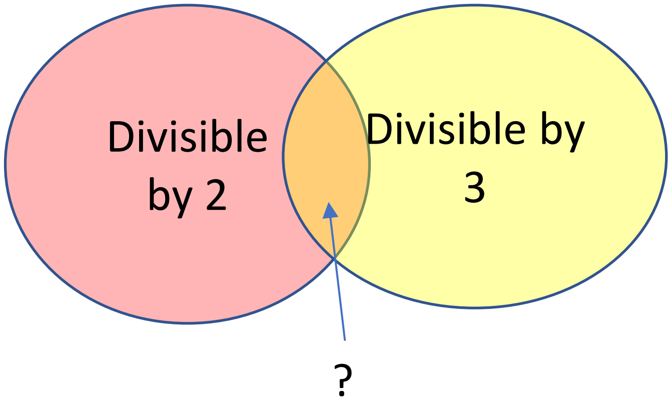
- How many positive integers from 1 to 10 are divisible by 2?
- How many positive integers from 1 to 10 are divisible by 3?
- How many are divisible by 2 or 3?

1	2	3	4	5	6	7	8	9	10

Exercise 1: counting integers

avoid double-counting

- How many positive integers from 1 to 10 are divisible by 2?
- How many positive integers from 1 to 10 are divisible by 3?
- How many are divisible by 2 or 3?



What is the probability that the next integer between 1 and 10 will be divisible by 2 or 3?

Counting rule #2: Rule of Product

- If we have 2 sets F and D, we can count all possible pairs of elements by multiplying |F|*|D|
- The size of Cartesian product |F x D|=|F|*|D|
- Example:
  - F: set of 5 food items
  - D: set of 7 drinks
  - How many ways to form a combo?

Rule of product for more than 2 sets

- F: set of 5 food items
- D: set of 7 drinks
- C: set of 4 cookie types

- How many combos now?

We can also pair items **from the same set**

- The rule of product still works
- Example:
  - How many 4-digit pins can be constructed from 10 available digits?
  - The digits can repeat

We can pair items from the same set

- The rule of product still works
- Example:
  - How many **4**-digit pins can be constructed from 10 available digits?

	For each first digit:	For each second digit:	For each third digit:	For each fourth digit:
1 of 10: total 10 choices	10 choices	10 choices	10 choices	10 choices

Total: 10*10*10*10 = $\mathbf{10^4}$ different pins

Selection: general formula

- How many groups of size k we can build by selecting from a set of n of available items?
- The same item can be selected more than once (with repetitions)

$$n^k$$

Different types of selection

	With repetitions	Without repetitions
Order matters	selection n^k ✓	
Order does not matter		

Generate different sequences of length k out of n available elements

There is an **unlimited supply** of elements to select from

Order matters: {a,b} is not the same as {b,a}

Exercise 2: hacker estimator

- How many different passwords of length at least 2 and at most 5 can be constructed from 26 letters?

Combining rule of product and rule of sum

- How many different passwords of length at least 2 and at most 5 can be constructed from 26 letters?

$$26^2 + 26^3 + 26^4 + 26^5$$

Sum of geometric series

$26^2 + 26^3 + 26^4 + 26^5$

- Each element in this sum is 26 times greater than the previous
- The sequence recursively defined as $\mathbf{a_n=a_{n-1}\cdot r}$ is called a *geometric progression* (or geometric series):

  $\{a, ar, ar^2, ar^3, \ldots\}$

- Our sequence starts from a=1, except that it is missing the first 2 elements

- The sum of geometric series is:

  $\mathbf{a*(1 - r^n)/(1 - r)}$

Proof by induction: https://www.ck12.org/book/ck-12-math-analysis/section/7.5/

How many passwords: easy compute

- How many different passwords of length at least 2 and at most 5 can be constructed from 26 letters?

$$26^2 + 26^3 + 26^4 + 26^5$$

- That reminds a part of the first 6 elements of geometric series with a=1 and r=26:

$$1,\ 1*26,\ 1*26^2,\ 1*26^3,\ 1*26^4,\ 1*26^5$$

- To sum these up we use the formula for the sum of the first n elements of geometric series:

$$\mathbf{a*(1 - r^n)/(1 - r)}$$

where a=1, r=26, n = 6

- In our case a=1, r=26, n = 6. The sum is $1*(1 - 26^6)/(1 - 26) = 12{,}356{,}631$
- We just need to subtract the first 2 elements to get the result:

12,356,631 – 1 – 26 = 12,356,604 distinct passwords

Bad password requirements!

- How many different passwords of length at least 2 and at most 5 can be constructed from 26 letters?

Answer: 12,356,631 possible passwords

- A supercomputer can perform 1 quintillion operations **each second**—a billion billions.
- Written out, that number looks like this: 1,000,000,000,000,000,000.

Exercise 3: enough license plates?

- The license plates contain 3 digits and 3 letters in the following fixed format:

A###AA

- How many different license plates can be created using 10 digits and 26 letters of English alphabet?

Different problem: rearranging 'candy'

- In how many ways can we rearrange letters in word 'candy'?

Permutations: permuting order

- In how many ways can we rearrange letters in word ‘candy’?

Select any of 5

Per each selection of the first – 4 for the second …

5	4	3	2	1

Total: 5*4*3*2*1 = 5! = 120

Selection without repetitions: permutations

	With repetitions	Without repetitions
Order matters	selection n^k	
Order does not matter		

Generate different sequences of length *k* out of n available elements

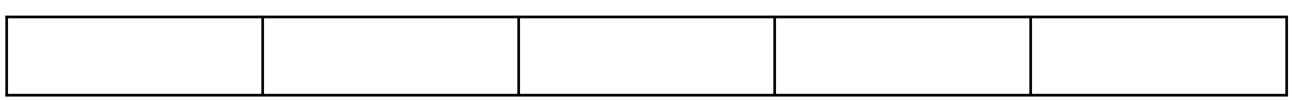
There is a **limited supply** of elements to select from (the same element can only be selected once)

The order still matters

More general example: selecting *k* items from *n* items without repetitions

- In how many different ways we can arrange n books on the shelf?
- The shelf **can only hold k books**: how many different shelf arrangements we can produce from n available books?

k = 5, n=10



Example: selecting *k* items from *n* items without repetitions

- In how many different ways we can arrange n books on the shelf?
- The shelf can only hold k books: how many different shelf arrangements we can produce from n available books?

k = 5, n=10

10	9	8	7	6

10*9*8*7*6 different ways

K-Permutations: general formula

- In how many different ways we can arrange n books on the shelf?
- The shelf can only hold k books: how many different shelves we can produce from n available books?

k = 5

n	n-1	n-2	n-3	n-4

Total: **P(n,k) = n(n-1)(n-2)…(n-(k-1))**

Can be represented as
n(n-1)(n-2)…(n-(k-1)) *(n-k)….1/ (n-k)….1
= n!/(n-k)!

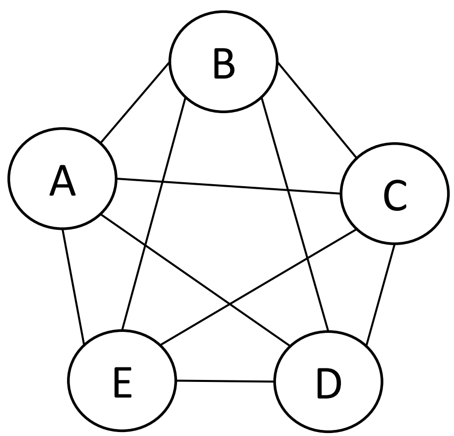
Works also for n=k
because 0! is defined to be 1

Selection **without** repetitions: order matters

	With repetitions	Without repetitions
Order matters	n^k ✔ selections	$n!/(n-k)!$ ✔ permutations
Order does not matter		

Different problem: tournament

- 5 teams played with each other
- How many games in total?



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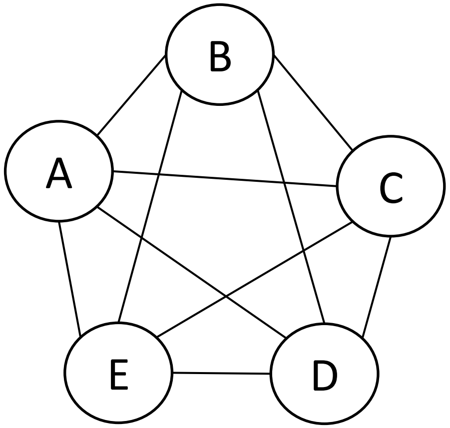

There are *n* choices of the first team in a game and (*n* - 1) choices of the second team

- Hence, there are n(n-1) = 5 * 4 = 20 games

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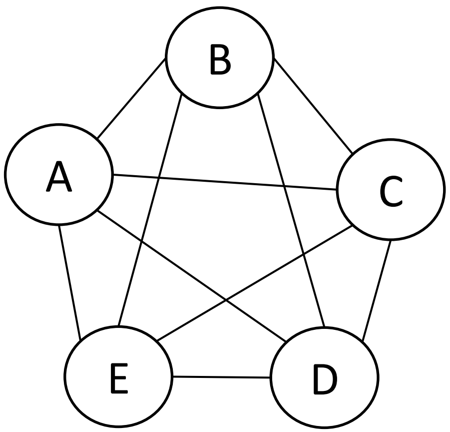

There are *n* choices of the first team in a game and (*n* - 1) choices of the second team

- Hence, there are n(n-1) = 5 * 4 = 20 games ?

Do not count the same object twice!

- The number of games in a tournament with n teams (each pair of teams played each other exactly once) is n(n - 1): there are *n* choices of the first team in a game and (*n* - 1) choices of the second team
- **Each game was counted twice:** the game between teams *i* and *j* was counted as *ij* and as *ji*
- Thus, the total number of games should be n(n - 1)/2

Selection without repetitions: order does not matter

	With repetitions	Without repetitions
Order matters	n^k	n!/(n-k)!
Order does not matter		

Generate different subsets of length k out of n available elements

There is a **limited supply** of elements to select from

The **order** in a subset **does not matter**: so subset a,b is exactly the same as b,a

Example: selecting passengers

- You have 5 friends and only 3 places in the car
- How many variations are there to invite your friends for a trip?

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5	4	3

Total: 5*4*3 = 60

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Because the order of friends does not matter, abc=bac=cab=…
We overcounted each sequence precisely by …

How many permutations each 3-element sequence may have?

Example: selecting passengers

- You have 5 friends and only 3 places in the car
- How many variations are there to invite your friends for a trip?

5	4	3

Total: 5*4*3 = 60

We need to divide the total by 3!

Selection of unordered subsets without repetitions: Combinations

- The number of ways to choose a subset of size k (when the order of elements does not matter) out of available n elements is:

$$\frac{n!}{(n-k)!k!} \qquad C(n,k) \qquad \textbf{n choose k} \qquad \binom{n}{k}$$

- The k! in the denominator normalizes all elements which are considered the same, because the order does not matter

Exercise 4: poker hands

- How many 5-card hands we can obtain from a standard 52-card deck?

Exercise 5: ways to reorder ‘*banana*’

- Combining knowledge of permutations and combinations

Exercise 5: ways to reorder *'banana'*

- Combining knowledge of permutations and combinations

In general: The number of distinguishable permutations that can be formed from a collection of n objects where the first object appears k_1 times, and the second object appears k_2 times, and so on, is:

$$n!/(k_1!*k_2!*\ldots*k_r!)$$

Selections so far

	With repetitions	Without repetitions
Order matters	n^k ✓ selections	n!/(n-k)! ✓ permutations
Order does not matter		n choose k ✓ combinations

Selection with repetitions: order does not matter

	With repetitions	Without repetitions
Order matters	n^k	n!/(n-k)!
Order does not matter		n choose k

Generate different subsets of length k out of n available elements

There is an **unlimited supply** of elements to select from: so there could be a,a and b,b

The **order** in a subset **does not matter**: so subset {a,b} is exactly the same as {b,a}

Example: small salad

- We have an unlimited supply of tomatoes, yellow peppers, and peas (n=3 types of ingredients to select from)
- We need to create a salad with a total of k=4 vegetables
- How many different salads we can create?

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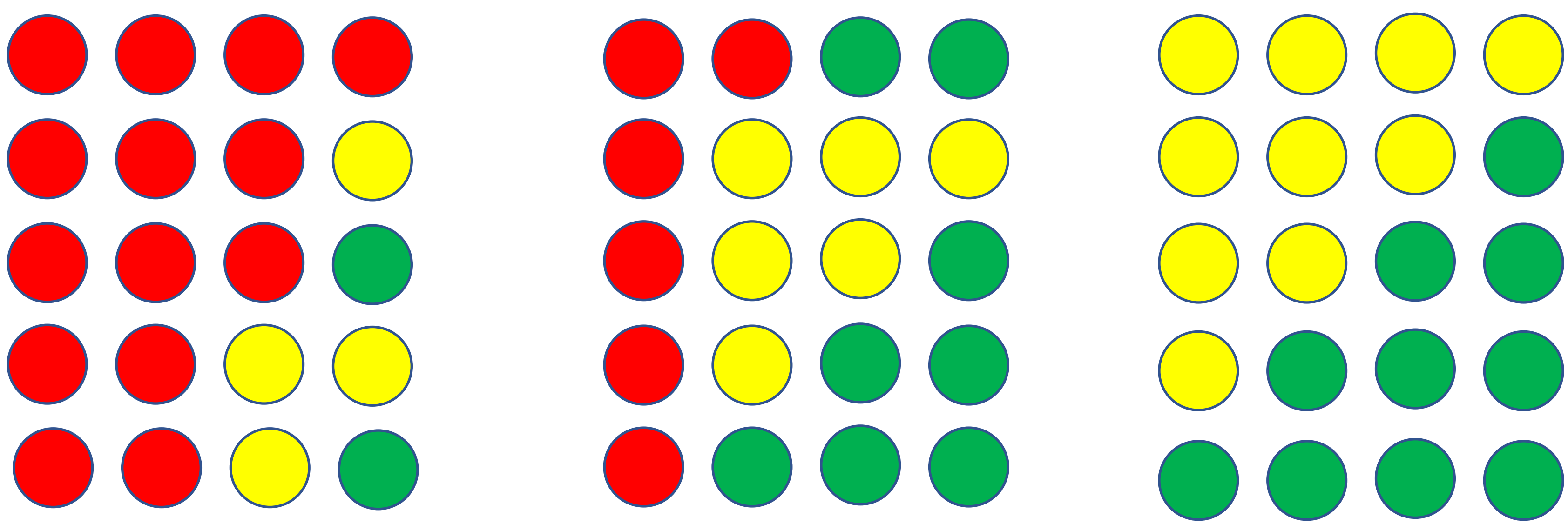
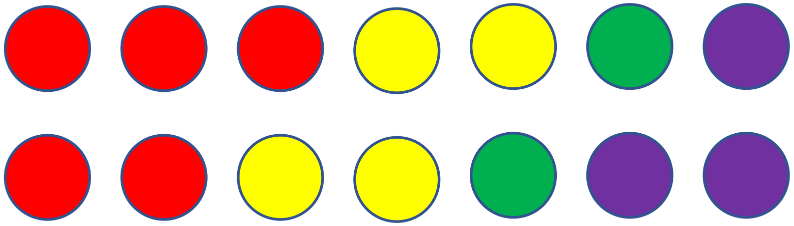


Example: big salad

- We have an unlimited supply of tomatoes, yellow peppers, peas, and eggplants: n = 4
- We need to create a salad by selecting k=7 vegetables
- How many different salads now?

Enumerating?



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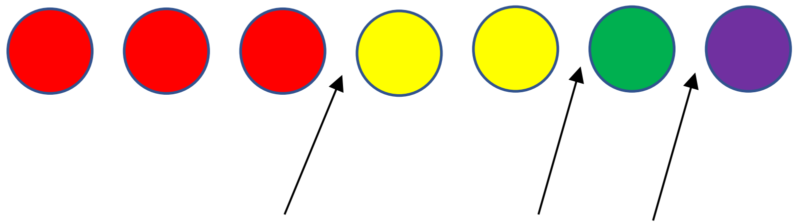
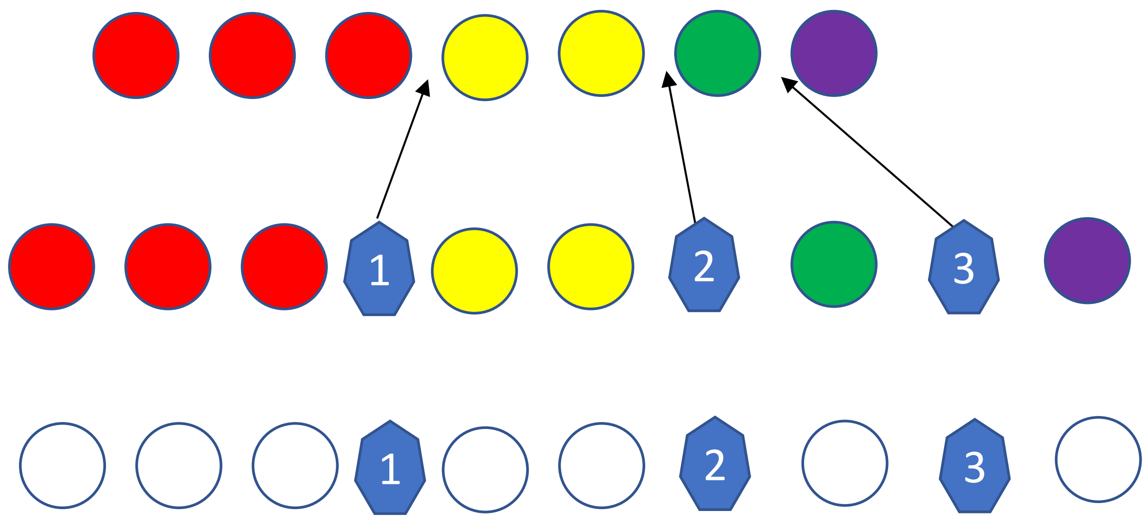
We really need a formula!

Solving salad



- If we agree to always keep the list of ingredients in certain order (tomatoes → peppers → peas → eggplants) then each configuration can be uniquely specified by delimiters between different types of elements
- If we defined the order, then we do not even need to mark the elements: they all become the same
- If we have n=4 types of elements – we need n-1=3 delimiters

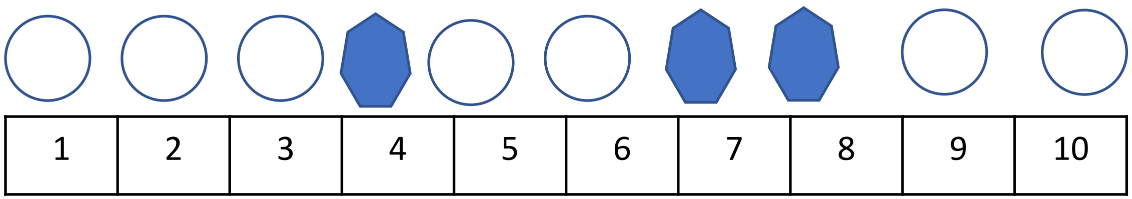
Solving salad



- Let’s make the delimiters into separate elements with a special meaning and add them to the original sequence of elements

- Note that this also works for configurations without peas, for example:

Solving salad



- Now we need to compute the number of distinct placements of n-1=3 delimiters into k + n - 1 = 10 available spots
- The order of delimiters does not really matter because we know the order of our ingredients

- So, we have (k + n - 1 = 7+3) different positions: 1,2,3,4, … 8,9,10
- How many ways are there to select n-1 = 3 distinct positions for dividers when the order does not matter?

Combinations with repetitions: general formula

- The number of combinations of size k of n objects with repetitions is equal to:

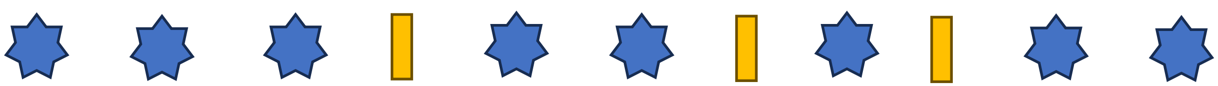
$$\binom{k+n-1}{n-1}$$

Where:
k is a size of the desired selection
n is the total number of distinct items to select from (with unlimited supply)

in this case *k* may be > *n*

This formula is also called *Stars and Bars*

- We think of items as stars
- The size of the section is k
- We use bars to signify delimiters for creating *n* groups: n-1 delimiters
- We arrange both *k* stars (items) and *n*-1 bars (delimiters) in a sequence



- The formula is $\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$
- It computes the number of ways to choose the positions for the bars (or stars) in a sequence of n + k - 1 slots.

Summary: selection

	With repetitions	Without repetitions
Order matters	**1** n^k	**2** P(n,k) = n!/(n-k)!
Order does not matter	**4** k + n - 1 choose n - 1	**3** C(n,k) = n choose k

The main difficulty –
to figure out the problem type!

Example 1

- Select a balanced committee of 4 people from 5 woman and 12 men

	With repetitions	Without repetitions
Order matters	n^k	n!/(n-k)!
Order does not matter	(k + n - 1) choose (n - 1)	n choose k

Example 1

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Example 2

- There are 15 identical candies.
- How many ways are there to distribute them among 7 kids?

	With repetitions	Without repetitions
Order matters	n^k	n!/(n-k)!
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Example 3

- How many 4-digit numbers with at least one digit 7?

	With repetitions	Without repetitions
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Example 3

- How many 4-digit numbers with at least one digit 7?

	With repetitions	Without repetitions
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Example 4

- How many 4-digit integers can be formed in such a way that the digits are strictly increasing:
- 1234, 1458 …

	With repetitions	Without repetitions
Order matters	n^k	n!/(n-k)!
Order does not matter	(k + n - 1) choose (n - 1)	n choose k

Example 4

- How many 4-digit integers can be formed in such a way that the digits are strictly increasing:
- 1234, 1458 …

	With repetitions	Without repetitions
Order matters	n^k	n!/(n-k)!
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Using counting techniques to compute probabilities of events

1. Count total number of ALL possible outcomes: N
2. Count number of outcomes where event A is true: T
3. To find P(A) divide T/N

Exercise 5. Balanced car renting agency

- The car renting agency has 30 cars in total:
  - 18 compact cars
  - 12 midsize cars

- If 4 cars are selected at random, what is the probability of getting 2 cars of each type?

Exercise 6. One defective tire

- The store has 20 tires, of each 3 are known to be defective.

- If you pick 4 tires at random: what is the probability to get precisely one of them defective?

Exercise 7. Royal Flush

- What is the probability of the Royal Flush in poker:
  A, K, Q, J, 10 - all the same suit

Sample problem 1: Birthday paradox

- What is the probability that in a room of 30 people there is a pair that shares the same birthday?

Sample problem 1: Birthday paradox

- What is the probability that in a room of 30 people there is a pair that shares the same birthday?

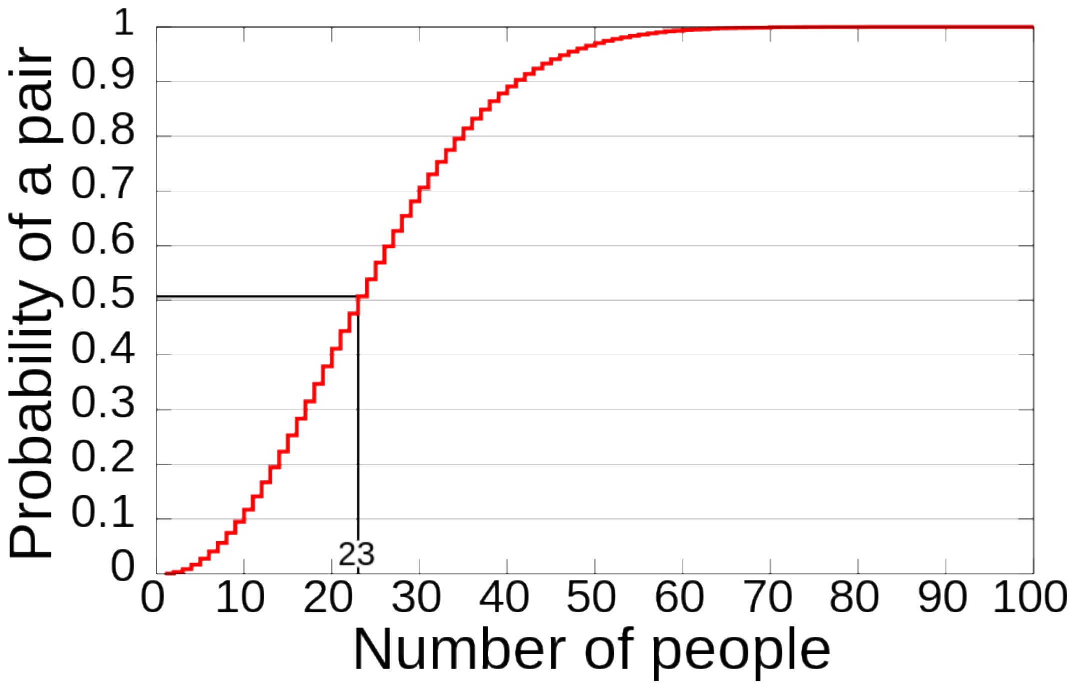
- The total number of ways to create a list of 30 birthdays from 365 possible days [order matters, days can repeat]:

N = 365^{30}

- Let ¬T be the total number of ways to select 30 days that are all different [[order matters, days cannot repeat]:

¬T = (365!)/(335!)

- Probability that all 30 are different: P(¬T)=¬T/N= 365*364*…336/365^{30}
- Probability that not all 30 are different: P(T) = 1 – P(¬T/A) = **0.71**



http://commons.wikimedia.org/wiki/File:Birthday_Paradox.svg

Sample problem 2: Same-day birthday

- Success T – in the room of 30 somebody has the same birthday as me

Sample problem 2: Same-day birthday

- Success T – in the room of 30 somebody has the same birthday as me

- Easier to find the probability of failure ¬T – everyone has birthday different from mine

- I have a birthday at one specific day of the year
- A - total number of all possible 29 birthdays: 365^{29}
- ¬T - total number of possible 29 birthdays selected from 364 days (different than mine): 364^{29}
- P(¬T) = ¬T/A= $364^{29}/365^{29}$

- The P (T) = 1 - P(¬T) = 0.0765