Binary relations establish a relationship between elements of two sets

Definition: Let A and B be two sets. A binary relation from A to B is a subset of $A \times B$.

In other words, a binary relation R is a set of ordered pairs (a_i, b_i) where $a_i \in A$ and $b_i \in B$.

Notation: We say that

- $a \, R \, b$ if $(a,b) \in R$
- $a \not{R} \, b$ if $(a,b) \notin R$

Definition: A relation on the set A is a relation from A to A. That is, a relation on the set A is a subset of $A \times A$.

What is an equivalence relation?

Informally: An equivalence relation partitions elements of a set into classes of “equivalent” objects.

Formally: A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

- reflexive: How can a relation define equivalent objects if an element isn’t equivalent to itself?
- symmetric: If x is equivalent to y, shouldn’t y be equivalent to x?
- transitive: If x is equivalent to y, and y is equivalent to z, shouldn’t x also be equivalent to z?

Definition: Two elements a and b that are related by some equivalence relation are called equivalent. We denote this by $a \sim b$ (or $a \sim_R b$).

Example: Comparing Magnitudes

Example: Let R be the relation on the set of integers such that a R b if and only if a = b or a = -b. Is R an equivalence relation?

Intuition says yes, so let's verify:

- Is R reflexive?
- Is R symmetric?
- Is R transitive?

Conclusion: Since R is symmetric, reflexive, and transitive, we know that R is an equivalence relation.

Congruence Modulo m

Example: Let m be a positive integer greater than 1. Show that R = {(a,b) | a ≡ b (mod m)} is an equivalence relation.

Solution:

- Recall: a ≡ b (mod m) ↔ m | (a - b)
- Is R reflexive?
  - a ≡ a (mod m) ↔ m | (a - a)
  - m | 0 since 0 = 0 × m
  - Yes, R is reflexive
- Is R symmetric?
  - If a ≡ b (mod m), then m | (a - b), so (a - b) = km for some k
  - Note that (b - a) = -km
  - So b ≡ a (mod m) and R is symmetric
- Is R transitive?
  - a ≡ b (mod m) means that (a-b) = km, so a = km + b
  - b ≡ c (mod m) means that (b-c) = jm, so c = b - jm
  - Note that a - c = (km + b) - (b - jm) = km + jm = (k+j)m
  - Since m | (a - c), a ≡ c (mod m), and R is transitive
- Conclusion: R is an equivalence relation

What about the "divides" relation?

Example: Is the "divides" relation on positive integers an equivalence relation?

Solution:

- Reflexive?
- Symmetric?
- Transitive?

- Conclusion: Since the "divides" relation is not symmetric, it cannot be an equivalence relation.

String Length

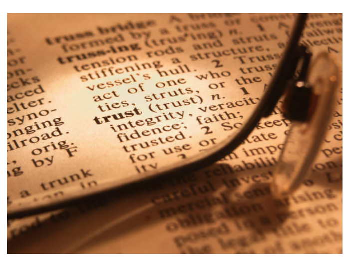
Example: Suppose that R is the relation on the set of strings of English letters such that a R b iff l(a) = l(b), where l(x) is the length of string x. Is R an equivalence relation?

Solution:

- Reflexive?
- Symmetric?
- Transitive?

R c.

- Conclusion: R is an equivalence relation



Magnitude of differences

Example: Let R be the relation on the set of real numbers such that x R y iff x and y are real numbers that differ by less than 1, i.e., |x – y| < 1. Is R an equivalence relation?

Solution:

- First, a few test cases:
  - 1.1 R 2.0? Yes, since
  - 1.1 R 3.0? No, since |
  - 2.0 R 2.5? Yes, since
- Reflexive?
- Symmetric?
- Transitive?
- Conclusion: Since R is not transitive, it cannot be an equivalence relation.

What is an equivalence class?

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to some element a is called the equivalence class of a.

Note: We denote the equivalence class of element a under relation R as $[a]_R$. If only one relation is being considered, we can drop the subscript and denote the equivalence class of a as [a].

Example: What are the equivalence classes of 0 and 1 under congruence modulo 4?

- [0] contains all integers x such that x ≡ 0 (mod 4)
- [1] contains all integers x such that x ≡ 1 (mod 4)
- So [0] = {…, -8, -4, 0, 4, 8, …}
- And [1] = {…, -7, -3, 1, 5, 9, …}

Variable names in C

Example: Some compilers for the C programming language truncate variable names after the first 31 characters. As a result, any two variable names that agree in the first 31 characters are considered to be identical. What are the equivalence classes of the variable names "Number_of_tropical_storms", "Number_of_named_tropical_storms", and "Number_of_named_tropical_storms_in_the_Atlantic_in_2005"?

Solution:

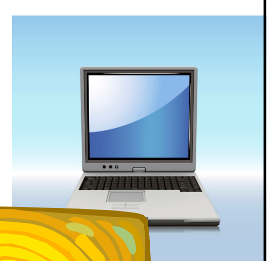
- [Number_of_tropical_storms] =
- [Number_of_named_tropical_storms] =
- [Number_of_named_tropical_storms_in_the_Atlantic_in_2005] =

An equivalence relation divides a set into disjoint subsets

(Contrived) Example: At State University, a student can either major in computer science or art history, but not both. Let R be the relation defined such that a R b if a and b are in the same major.

Observations:

- R is an equivalence relation (Why?)
- R breaks the set S of all students into two subsets:
  - C = Students majoring in computer science
  - A = Students majoring in art history
- No student in C is also in A
- No student in A is also in C
- C and A are equivalence classes of S



Equivalence classes are either equal or disjoint

Theorem: If R is an equivalence relation on some set A, then the following three statements are equivalent: (i) a R b, (ii) [a] = [b], and (iii) [a] ∩ [b] ≠ ∅.

Proof:

- To prove this, we'll prove that (i) → (ii), (ii) → (iii), and (iii) → (i)
- (i) → (ii)
  - Assume that a R b
  - To prove that [a] = [b], we will show that [a] ⊆ [b] and [b] ⊆ [a]
  - Suppose that c ∈ [a], then a R c
  - Since a R b and R is symmetric, we have that b R a
  - Since R is transitive, we have that b R a and a R c, so b R c
  - This means that c ∈ [b] and thus that [a] ⊆ [b]
  - The proof that [b] ⊆ [a] is identical

Equivalence classes are either equal or disjoint

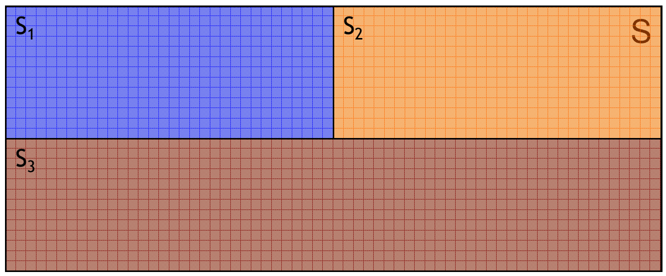
Theorem: If R is an equivalence relation on some set A, then the following three statements are equivalent: (i) a R b, (ii) [a] = [b], and (iii) [a] ∩ [b] ≠ ∅.

Proof (cont.):

- (ii) → (iii)
  - Assume that [a] = [b]
  - [a] ∩ [b] is non-empty since a ∈ [a]
- (iii) → (i)
  - Assume that [a] ∩ [b] ≠ ∅
  - This means that there exists some element c ∈ [a] ∩ [b]
  - So, a R c and b R c
  - By symmetry, we have that c R b
  - By transitivity, we have that a R c and c R b means a R b
- Since (i) → (ii), (ii) → (iii), and (iii) → (i), all three statements are equivalent. ❑

Equivalence classes partition a set

Definition: A partition of a set S is a collection of disjoint subsets that have S as their union.



Observation: The equivalence classes of a set partition that set.

- $\cup_{a \in A}$ [a] = A since each a ∈ A is in its own equivalence class
- By our theorem, we know that either [a] = [b], or [a] ∩ [b] = ∅

The integers (mod m), redux

Example: What are the sets in the partition produced by the equivalence relation equivalence mod 4?

Solution:

- [0] = {…, -8, -4, 0, 4, 8, …}
- [1] = {…, -7, -3, 1, 5, 9, …}
- [2] = {…, -6, -2, 2, 6, 10, …}
- [3] = {…, -5, -1, 3, 7, 11, …}

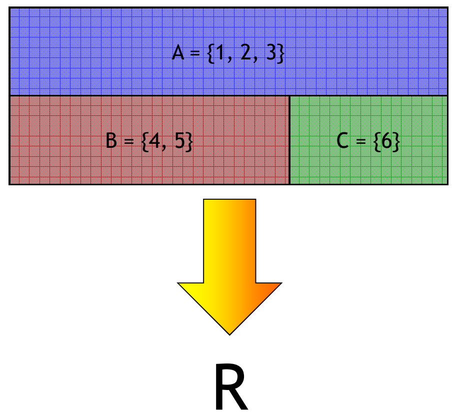
- Note that each integer is in one of these sets, and each set is disjoint. Thus, these equivalence classes partition the set **Z**.

Conversely, a partition of a set describes an equivalence relation

Example: List the ordered pairs in the equivalence relation R produced by the partition A = {1,2,3}, B={4,5}, C={6} of S={1,2,3,4,5,6}.

Solution:

- From A = {1,2,3} we have
  - (1,1), (1,2), (1,3) ∈ R
  - (2,1), (2,2), (2,3) ∈ R
  - (3,1), (3,2), (3,3) ∈ R
- From B = {4,5} we have
  - (4,4), (4,5) ∈ R
  - (5,4), (5,5) ∈ R
- From C = {6} we have
  - (6,6) ∈ R



Group Work!

Problem 1: Which of the following relations on {0, 1, 2, 3} are equivalence relations? Which properties are lacking from those relations that are not equivalence relations?

1. **{(0, 0), (1,1), (2,2), (3,3)}**
2. **{(0,0), (0,2), (2, 0), (2,2), (2,3), (3,2), (3,3)}**

Problem 2: Which of these collections of sets are partitions of the set S = {1,2,3,4,5,6}?

1. **{1,2}, {2,3,4}, {4,5,6}**
2. **{2,4,6}, {1,3,5}**
3. **{1,4,5}, {2, 6}**