Today

Relations

- Binary relations and properties
- Relationship to functions

n-ary relations

- Definitions

Binary relations establish a relationship between elements of two sets

Definition: Let A and B be two sets. A binary relation from A to B is a subset of A × B.

In other words, a binary relation R is a set of ordered pairs (a_i, b_i) where $a_i \in A$ and $b_i \in B$.

Notation: We say that

- a R b if $(a,b) \in R$
- a $\not R$ b if $(a,b) \notin R$

Example: Course Enrollments

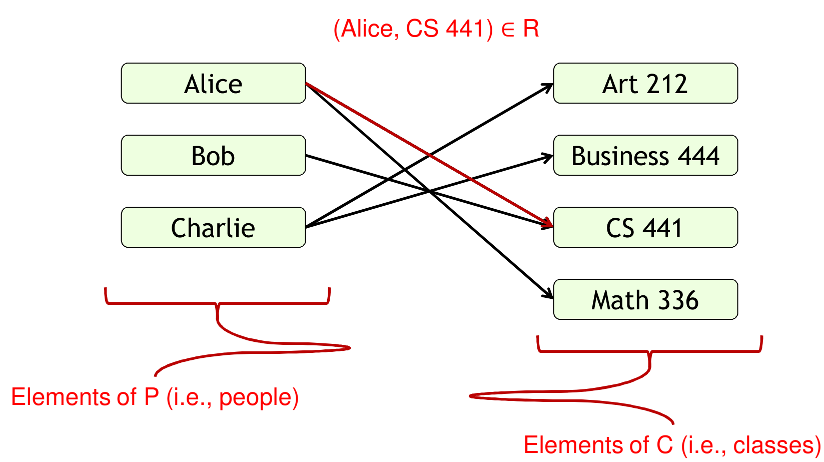
Let's say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation R that represents the relationship between people and classes.

Solution:

- Let the set P denote people, so P = {Alice, Bob, Charlie}
- Let the set C denote classes, so C = {CS 441, Math 336, Art 212, Business 444)
- By definition $R \subseteq P \times C$
- From the above statement, we know that
  - (Alice, CS 441) ∈ R
  - (Bob, CS 441) ∈ R
  - (Alice, Math 336) ∈ R
  - (Charlie, Art 212) ∈ R
  - (Charlie, Business 444) ∈ R
- So, R = {(Alice, CS 441), (Bob, CS 441), (Alice, Math 336), (Charlie, Art 212), (Charlie, Business 444)}

A relation can also be represented as a graph

Let's say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation R that represents the relationship between people and classes.



A relation can also be represented as a table

Let's say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation R that represents the relationship between people and classes.

Name of the relation

Elements of C (i.e., courses)

(Bob, CS 441) ∈ R

R	Art 212	Business 444	CS 441	Math 336
Alice			X	X
Bob			X	
Charlie	X	X		

Elements of P (i.e., people)

Wait, doesn't this mean that relations are the same as functions?

Not quite… Recall the following definition from Lecture #9.

Definition: Let A and B be nonempty sets. A function, f, is an assignment of exactly one element of set B to each element of set A.

This would mean that, e.g., a person only be enrolled in one course!

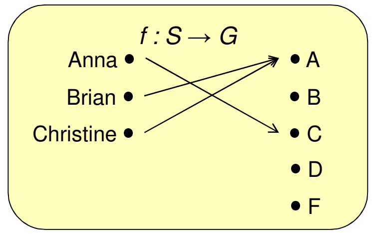
Reconciling this with our definition of a relation, we see that

1. Every function is also a relation
2. Not every relation is a function

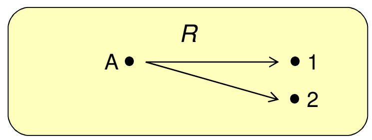
Let's see some quick examples…

Short and sweet…

1. Consider $f : S \rightarrow G$
   - Clearly a function
   - Can also be represented as the relation R = {(Anna, C), (Brian, A), (Christine A)}



1. Consider the set R = {(A, 1), (A, 2)}
   - Clearly a relation
   - Cannot be represented as a function!



We can also define binary relations on a single set

Definition: A relation on the set A is a relation from A to A. That is, a relation on the set A is a subset of A × A.

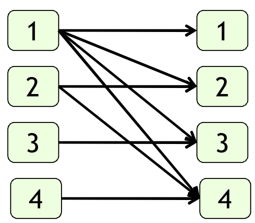
Example: Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?

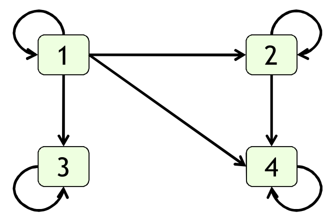
Solution:

- 1 divides everything
- 2 divides itself and 4
- 3 divides itself
- 4 divides itself
- So, R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

Representing the last example as a graph…

Example: Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?





Tell me what you know…

Question: Which of the following relations contain each of the pairs (1,1), (1,2), (2,1), (1,-1), and (2,2)?

- $R_1 = \{(a,b) \mid a \leq b\}$
- $R_2 = \{(a,b) \mid a > b\}$
- $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a,b) \mid a = b\}$
- $R_5 = \{(a,b) \mid a = b + 1\}$
- $R_6 = \{(a,b) \mid a + b \leq 3\}$

These are all relations on an infinite set!

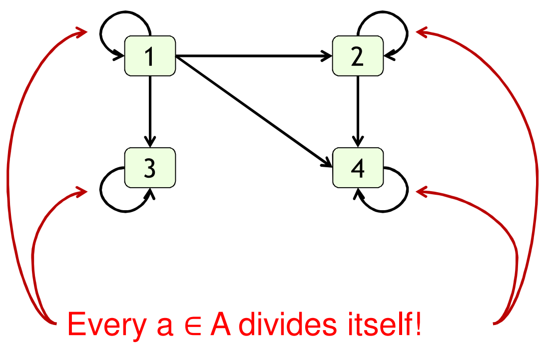
Answer:

	(1,1)	(1,2)	(2,1)	(1,-1)	(2,2)
R_1					
R_2					
R_3					
R_4					
R_5					
R_6					

Properties of Relations

Definition: A relation R on a set A is reflexive if (a,a) ∈ R for every a ∈ A.

Note: Our "divides" relation on the set A = {1,2,3,4} is reflexive.



	1	2	3	4
1	X	X	X	X
2		X		X
3			X	
4				X

Properties of Relations

Definition: A relation R on a set A is symmetric if (b,a) ∈ R whenever (a,b) ∈ R for every a,b ∈ A. If R is a relation in which (a,b) ∈ R and (b,a) ∈ R implies that a=b, we say that R is antisymmetric.

Mathematically:

- Symmetric: $\forall a \forall b((a,b) \in R \rightarrow (b,a) \in R)$
- Antisymmetric: $\forall a \forall b(((a,b) \in R \wedge (b,a) \in R) \rightarrow (a = b))$

Examples:

- Symmetric: R = {(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)}
- Antisymmetric: R = {(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)}

Symmetric and Antisymmetric Relations

R = {(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)}

	1	2	3	4
1	X	X	X	X
2	X		X	
3	X	X		
4	X			X

Symmetric relation

- Diagonal axis of symmetry
- Not all elements on the axis of symmetry need to be included in the relation

R = {(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)}

	1	2	3	4
1	X	X	X	X
2				X
3			X	
4				X

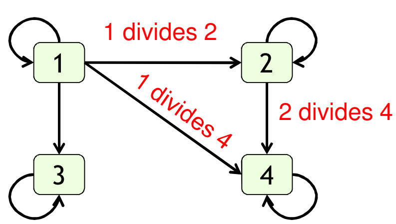
Asymmetric relation

- No axis of symmetry
- Only symmetry occurs on diagonal
- Not all elements on the diagonal need to be included in the relation

Properties of Relations

Definition: A relation R on a set A is transitive if whenever (a,b) ∈ R and (b,c) ∈ R, then (a,c) ∈ R for every a,b,c ∈ A.

Note: Our "divides" relation on the set A = {1,2,3,4} is transitive.



This isn't terribly interesting, but it is transitive nonetheless….

More common transitive relations include equality and comparison operators like <, >, ≤, and ≥.

Examples, redux

Question: Which of the following relations are reflexive, symmetric, antisymmetric, and/or transitive?

- $R_1 = \{(a,b) \mid a \le b\}$
- $R_2 = \{(a,b) \mid a > b\}$
- $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a,b) \mid a = b\}$
- $R_5 = \{(a,b) \mid a = b + 1\}$
- $R_6 = \{(a,b) \mid a + b \le 3\}$

Answer:

	Reflexive	Symmetric	Antisymmetri	Transitive
R_1	[illegible]	[illegible]	[illegible]	[illegible]
R_2	[illegible]	[illegible]	[illegible]	[illegible]
R_3	[illegible]	[illegible]	[illegible]	[illegible]
R_4	[illegible]	[illegible]	[illegible]	[illegible]
R_5	[illegible]	[illegible]	[illegible]	[illegible]
R_6	No	Yes	No	No

Relations can be combined using set operations

Example: Let R be the relation that pairs students with courses that they have taken. Let S be the relation that pairs students with courses that they need to graduate. What do the relations $R \cup S$, $R \cap S$, and $S - R$ represent?

Solution:

- $R \cup S$ = All pairs (a,b) where
  - student a has taken course b OR
  - student a needs to take course b to graduate
- $R \cap S$ = All pairs (a,b) where
  - Student a has taken course b AND
  - Student a needs course b to graduate
- $S - R$ = All pairs (a,b) where
  - Student a needs to take course b to graduate BUT
  - Student a has not yet taken course b

Relations can be combined using functional composition

Definition: Let R be a relation from the set A to the set B, and S be a relation from the set B to the set C. The composite of R and S is the relation of ordered pairs (a, c), where a ∈ A and c ∈ C for which there exists an element b ∈ B such that (a, b) ∈ R and (b, c) ∈ S. We denote the composite of R and S by R ° S.

Example: What is the composite relation of R and S?

R: {1,2,3} → {1,2,3,4}

- R = {(1,1),(1,4),(2,3),(3,1),(3,4)}

S: {1,2,3,4} → {0,1,2}

- S = {(1,0),(2,0),(3,1),(3,2),(4,1)}

So: R ° S = {(1,0), (3,0), (1,1), (3,1), (2,1), (2,2)}

Group Work!

Problem 1: List the ordered pairs of the relation R from A = {0,1,2,3,4} to B = {0,1,2,3} where (a,b) ∈ R iff a + b = 4.

Problem 2: Is the relation {(2,4), (4,2)} on the set {1,2,3,4} reflexive, symmetric, antisymmetric, and/or transitive?

We can also "relate" elements of more than two sets

Definition: Let A_1, A_2, …, A_n be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets A_1, A_2, …, A_n are called the domains of the relation, and n is its degree.

Example: Let R be the relation on $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^+$ consisting of triples (a, b, m) where $a \equiv b \pmod{m}$.

- What is the degree of this relation?
- What are the domains of this relation?
- Are the following tuples in this relation?
  - (8,2,3)
  - (-1,9,5)
  - (11,0,6)

Final Thoughts

- Relations allow us to represent and reason about the relationships between sets in a more general way than functions did