Homework 9

Minimum Value	0.00
Maximum Value	100.00
Average	82.18
Median	88.00

90 – 100	20
80 – 89	15
70 – 79	5
60 – 69	2
0	3

Snow Makeup?

No need to meet on Saturday

How can we incorporate prior knowledge?

Sometimes we want to know the probability of some event given that another event has occurred.

Example: A fair coin is flipped three times. The first flip turns up tails. Given this information, what is the probability that an odd number of tails appear in the three flips?

Solution:

- Let F = "the first flip of three comes up tails"
- Let E = "tails comes up an odd number of times in three flips"
- Since F has happened, S is reduced to {THH, THT, TTH, TTT}
- By Laplace's definition of probability, we know:
- p(E) = |E|/|S|
- = |{THH, TTT}| / |{THH, THT, TTH, TTT}|
- = 2/4
- = 1/2

Conditional Probability

Definition: Let E and F be events with p(F) > 0. The conditional probability of E given F, denoted p(E | F), is defined as:

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

Intuition:

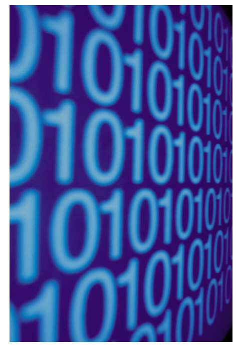
- Think of the event F as reducing the sample space that can be considered
- The numerator looks at the likelihood of the outcomes in E that overlap those in F
- The denominator accounts for the reduction in sample size indicated by our prior knowledge that F has occurred

Bit strings

Example: Suppose that a bit string of length 4 is generated at random so that each of the 16 possible 4-bit strings is equally likely to occur. What is the probability that it contains at least two consecutive 0s, given that the first bit in the string is a 0?

Solution:

- Let E = "A 4-bit string has at least two consecutive zeros"
- Let F = "The first bit of a 4-bit string is a zero"
- Want to calculate p(E | F) = p(E ∩ F)/p(F)
- E ∩ F = {0000, 0001, 0010, 0011, 0100}
- So, p(E ∩ F) = 5/16
- Since each bit string is equally likely to occur, p(F) = 8/16 = 1/2
- So p(E | F) = (5/16)/(1/2) = 10/16 = 5/8



Kids

Example: What is the conditional probability that a family with two kids has two boys, given that they have at least one boy? Assume that each of the possibilities BB, BG, GB, GG is equally likely to occur.

Boy is older (BG) **Girl is older** (GB)

Solution:

- Let E = "A family with 2 kids has 2 boys"
- E = {BB}
- Let F = "A family with 2 kids has at least 1 boy"
- F = {BB, BG, GB}
- E ∩ F = {BB}
- So p(E | F) = p(E ∩ F)/p(F)
- = (1/4) / (3/4)
- = 1/3

Does prior knowledge always help us?

Example: Suppose a fair coin is flipped twice. Does knowing that the coin comes up tails on the first flip help you predict whether the coin will be tails on the second flip?

Solution:

- S = {HH, HT, TH, TT}
- F = "Coin was tails on the first flip" = {TH, TT}
- E = "Coin is tails on the second flip" = {TT, HT}
- p(E) = 2/4 = 1/2
- p(E | F) = p(E ∩ F)/p(F)
- = (1/4) / (2/4)
- = 1/2
- Knowing the first flip does not help you guess the second flip!

Group Work!

Problem 1: What is the probability of these events when we randomly select a permutation of {1, 2, 3}?

- 1 precedes 3
- 3 precedes 1 and 3 precedes 2

Problem 2: What is the conditional probability that exactly 4 heads appear when a fair coin is flipped five times, given that the first flip came up heads?

Independent Events

Definition: We say that events E and F are independent if and only if p(E ∩ F) = p(E)p(F).

Recall: In our last example…

- S = {HH, HT, TH, TT}
- F = {TH, TT}
- E = {HT, TT}
- E ∩ F = {TT}

This checks out!

So:

- p(E ∩ F) = |E ∩ F|/|S|
- = 1/4
- p(E)p(F) = 1/2 × 1/2
- = 1/4

Example: Bit Strings

Example: Suppose that E is the event that a randomly generated bit string of length four begins with a 1, and F is the event that this bit string contains an even number of 1s. Are E and F independent if all 4-bit strings are equally likely to occur?

Solution:

- By the product rule, |S| = 2^4 = 16
- E = {1111, 1110, 1101, 1011, 1100, 1010, 1001, 1000}
- F = {0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111}
- So p(E) = p(F) = 8/16 = 1/2
- p(E)p(F) = 1/4
- E ∩ F = {1111, 1100, 1010, 1001}
- p(E ∩ F) = 4/16 = 1/4
- Since p(E ∩ F) = p(E)p(F), E and F are independent events

Example: Distribution of kids

Example: Assume that each of the four ways that a family can have two children are equally likely. Are the events E that a family with two children has two boys, and F that a family with two children has at least one boy independent?

Solution:

- E = {BB}
- F = {BB, BG, GB}
- p(E) = 1/4
- p(F) = 3/4
- p(E)p(F) = 3/16
- $E \cap F$ = {BB}
- $p(E \cap F)$ = 1/4
- Since 1/4 ≠ 3/16, E and F are not independent

Group Work!

Problem 1: Let E and F be the events that a family of 4 children has children of both sexes and has at most one boy, respectively. Are E and F independent?

If probabilities are independent, we can use the product rule to determine the probabilities of combinations of events

Example: What is the probability of flipping heads 4 times in a row using a fair coin?

Answer: p(H) = 1/2, so p(HHHH) = $(1/2)^4$ = 1/16

Example: What is the probability of rolling the same number 3 times in a row using an unbiased 6-sided die?

Answer:
- First roll agrees with itself with probability 1
- 2nd roll agrees with first with probability 1/6
- 3rd roll agrees with first two with probability 1/6
- So probability of rolling the same number 6 times is $1 \times 1/6 \times 1/6 = 1/36$

Final Thoughts

- More Probability Theory
  - Conditional probability
  - Independence