

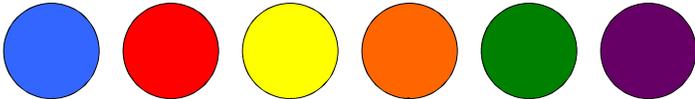
## Today's Topics

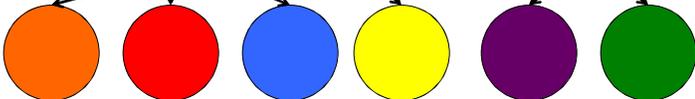


- Permutations
- Combinations
- Binomial coefficients

A permutation is an **ordered** arrangement of a set of objects



$S =$  

$\sigma(S) =$  

**Note:** A permutation of some set is essentially just a shuffle of that set.

## Sometimes we're interested in counting the number of ways that a given set can be arranged



**Example:** Suppose that a photographer wants to take a picture of three dogs. How many ways can the dogs be arranged?



There are six possible arrangements of three dogs!

## Counting permutations



In general, we can use the **product rule** to count the number of permutations of a given set.

Given a set of  $n$  items, we have:

- $n$  ways to pick the 1<sup>st</sup> item in the permuted set
- $n-1$  ways to pick the 2<sup>nd</sup> item in the permuted set
- $n-2$  ways to pick the 3<sup>rd</sup> item in the permuted set
- ...
- 1 way to choose the last item in the permuted set

So, for a set of size  $n$ , we have  $n \times (n-1) \times (n-2) \times \dots \times 1 = n!$  ways to permute that set

## And the winner is...



**Example:** Six friends run in a foot race. How many possible outcomes of the race are there, assuming that there are no ties?

**Solution:**

- 6 ways to choose 1<sup>st</sup> place
- 5 ways to choose 2<sup>nd</sup> place
- 4 ways to choose 3<sup>rd</sup> place
- 3 ways to choose 4<sup>th</sup> place
- 2 ways to choose 5<sup>th</sup> place
- 1 way to choose last place
- So there are  $6! = 720$  possible outcomes!

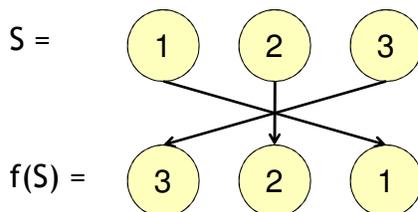


## Functions and permutations



Let  $S$  be some set. If  $f: S \rightarrow S$  is a bijection, then  $f$  describes a permutation of the set  $S$ .

**Example:** Let  $S = \{1, 2, 3\}$ ,  $f(1) = 3$ ,  $f(2) = 2$ ,  $f(3) = 1$



## More often than not, we're only interested in arranging a **subset** of a given set



**Definition:** An **r-permutation** is a permutation of some  $r$  elements of a set.

**Example:** Let  $S = \{\text{Alice, Bob, Carol, Dave}\}$ . Then:

- Dave, Bob is a **2-permutation** of  $S$
- Carol, Alice, Bob is a **3-permutation** of  $S$
- Bob, Dave is a **2-permutation** of  $S$

Rather than specifying a particular  $r$ -permutation of a set, we're usually more interested in **counting** the number of  $r$ -permutations of a set

## Counting $r$ -permutations



**Definition:** We denote the number of  $r$ -permutations of a set of size  $n$  by  **$P(n, r)$** . By the product rule,  $P(n, r) = n \times (n-1) \times \dots \times (n-r+1) = n!/(n-r)!$

**Example:** In a foot race between six people, how many ways can the gold, silver, and bronze medals be assigned, assuming that there are no ties?

**Solution:**

- $P(6, 3) = 6!/(6-3)! = 6!/3! = 720/6 = 120$
- So, 120 ways to assign medals



## The traveling salesperson

**Example:** A salesperson must visit 7 different cities. The first and last cities of her route are specified by her boss, but she can choose the order of the other visits. How many possible trips can she take?

**Solution:**

- Since first and last cities are fixed, we must count the number of ways to permute the 5 remaining cities
- So, there are  $5! = 120$  possible trips that the salesperson can take.



## Counting strings

**Example:** How many permutations of “ABCDEFGG” contain the substring “ABC”?

**Solution:**

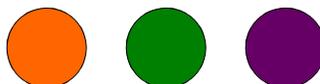
- Observation: Treat “ABC” as one character
- Now, how many ways are there to permute {ABC, D, E, F, G}?
- $|\{ABC, D, E, F, G\}| = 5$
- $5! = 120$  permutations contain the substring “ABC”



## How do we count **unordered** selections of objects?



**Example:** How many ways can we choose two objects from the set  $S =$



$$S_1 = \begin{array}{c} \text{orange} \\ \text{green} \end{array} = \begin{array}{c} \text{green} \\ \text{orange} \end{array}$$

$$S_2 = \begin{array}{c} \text{orange} \\ \text{purple} \end{array} = \begin{array}{c} \text{purple} \\ \text{orange} \end{array}$$

$$S_3 = \begin{array}{c} \text{green} \\ \text{purple} \end{array} = \begin{array}{c} \text{purple} \\ \text{green} \end{array}$$

**Conclusion:** There are three **2-combinations** of a set of size three.

## Counting r-combinations



**Definition:** Let  $C(n, r)$  denote the number of r-combinations of a set of size  $n$ . Then:

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

**Proof:**

- The r-permutations of the set can be formed by finding all of the r-combinations of the set and then permuting each r-combination in each possible way
- There are  $P(r, r)$  ways to permute each possible r-combination
- So,  $P(n, r) = C(n, r) \times P(r, r)$
- This means that  $C(n, r) = P(n, r) / P(r, r)$
- $= [n! / (n-r)!] / r!$
- $= n! / [r!(n-r)!] \quad \square$

## Alternate notation



**Note 1:** Sometimes,  $C(n, r)$  is read “n choose r”



**This is intuitive, as  $C(n, r)$  specifies the number of ways to choose  $r$  objects from a set of size  $n$ .**

**Note 2:**  $C(n, r)$  is often written as  $\binom{n}{r}$ . In this class, I will use the notation  $C(n, r)$  exclusively.

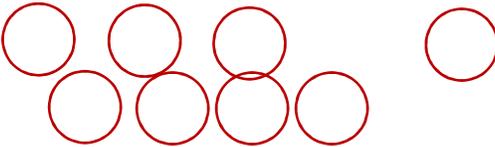
## Be careful using the formula for $C(n, r)$



Using the formula for  $C(n, r)$  directly can result in doing **lots** of multiplication...

Instead, note that much of the denominator cancels out terms in the numerator. For example:

$C(52, 5)$





## Choosing participants

**Example:** Say that a class consists of 30 students. How many ways can 3 people be chosen from this class to participate in a survey?

**Solution:**

- Want to compute is “30 choose 3”
- $C(30, 3) = 30!/[3!27!]$
- $= (30 \times 29 \times 28)/(3 \times 2)$
- $= 10 \times 29 \times 14$
- $= 4,060$  ways to choose participants



## Poker Hands

**Example:** Given a standard 52-card deck, how many 5 card poker hands can be drawn?

**Solution:**

- $C(52, 5) = 52!/[5!47!]$
- $= (52 \times 51 \times 50 \times 49 \times 48)/(5 \times 4 \times 3 \times 2 \times 1)$
- $= 26 \times 17 \times 10 \times 49 \times 12$
- $= 2,598,960$  different hands





## An interesting observation...

**Note:** Given a set of size  $n$ , **choosing**  $r$  elements is the same as **excluding**  $n-r$  elements.

This means that  $C(n, r) = C(n, n-r)$ .

**Proof:**

- $C(n, n-r) = n! / [(n-r)!(n-(n-r))!]$
- $= n! / [(n-r)!r!]$  by simplification
- $= n! / [r!(n-r)!]$  by the commutative property
- $= C(n, r)$  by definition  $\square$



## Permutations and combinations can be used in conjunction with the product and sum rules!

**Example:** Suppose the CS department has 11 faculty members and the Math department has 9 faculty members. How many ways can a committee consisting of 4 CS faculty members and 3 Math faculty members be chosen?

**Solution:**

- $C(11, 4)$  ways to choose 4 CS profs
- For each of these, there are  $C(9, 3)$  ways to choose 3 Math profs
- So, we need to compute  $C(11, 4) \times C(9, 3)$
- $C(11, 4) \times C(9, 3) = 11! / (4!7!) \times 9! / (3!6!)$
- $= 84 \times 330$
- $= 27,720$  ways!



## Group Work!

**Problem 1:** Consider a boat race with 12 entrants. How many ways can the 1<sup>st</sup>, 2<sup>nd</sup>, and 3<sup>rd</sup> place trophies be awarded?

**Problem 2:** How many sets of five letters be selected from the English alphabet?



## Combinations can be helpful when examining binomial expressions

**Definition:** A **binomial** is an expression involving the sum of two terms. For example  $(x + y)$  is a binomial, as is  $(3 + j)^4$ .

r-Combinations are also called **binomial coefficients**, since they occur as coefficients in the expansions of binomial expressions.

**Example:**

$$\begin{aligned}(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ &= C(3,0)x^3y^0 + C(3,1)x^2y^1 + C(3,2)x^1y^2 + C(3,3)x^0y^3\end{aligned}$$



## The Binomial Theorem

**The Binomial Theorem:** Let  $x$  and  $y$  be variables, and let  $n$  be a non-negative integer. Then:

$$(x + y)^n = \sum_{j=0}^n C(n, j)x^{n-j}y^j$$

**Example:** Compute  $(x + y)^2$ .

- $C(2, 0) = 1$
- $C(2, 1) = 2$
- $C(2, 2) = 1$
- $(x + y)^2 = x^2y^0 + 2x^1y^1 + x^0y^2$
- $= x^2 + 2xy + y^2$  ✓



## An easy example...

**Question:** What is the coefficient of  $x^{12}y^{13}$  in the expansion of  $(x+y)^{25}$ ?

**Solution:** By the binomial theorem, we know that this term can be written as  $C(25, 13) x^{12}y^{13}$ , so the coefficient of  $x^{12}y^{13}$  is  $C(25, 13) = 5,200,300$ .

## Using Pascal's identity to define $C(n, r)$ recursively



**Pascal's identity:**  $C(n+1, k) = C(n, k-1) + C(n, k)$   
(proof in book)

**Basis step:** For all  $n$ , we have that  $C(n, 0) = C(n, n) = 1$

**Recursive step:**  $C(n+1, k) = C(n, k-1) + C(n, k)$

**Example:** Compute  $C(3, 2)$ .

- $C(3, 2) = C(2, 1) + C(2, 2)$  ✓
- $= C(1, 0) + C(1, 1) + 1$
- $= 1 + 1 + 1$
- $= 3$
- $= 3! / (2!(3-2)!)$

## Group Work!



**Problem 1:** What is the coefficient of  $x^9$  in  $(2-x)^{19}$ ?



## Final Thoughts

- Permutations count the ways that we can shuffle a set.  $r$ -Permutations count the number of ways that we can shuffle  $r$  items in a set.
- $r$ -Combinations are useful when we want to count unordered subsets of a given set