Today's Topic: Propositional Logic

- What is a proposition?
- Logical connectives and truth tables
- Translating between English and propositional logic

Logic is the basis of all mathematical and analytical reasoning

Given a collection of known truths, logic allows us to deduce new truths

Example

Base facts:

If it is raining, I will not go outside
If I am inside, Stephanie will come over
Stephanie and I always play Scrabble if we are together during the weekend
Today is rainy Saturday

Conclusion: Stephanie and I will play Scrabble today

Logic allows us to advance mathematics through an iterative process of conjecture and proof

Propositional logic is a very simple logic

Definition: A proposition is a precise statement that is either true or false, but not both.

Examples:

- 2 + 2 = 4 (true)
- All dogs have 3 legs (false)
- $x^2 < 0$ (false)
- Washington, D.C. is the capital of the USA (true)

Not all statements are propositions

- Marcia is pretty
  - “Pretty” is a subjective term.
- $x^3 < 0$
  - True if $x < 0$, false otherwise.
- Springfield is the capital
  - True in Illinois, false in Massachusetts.

We can use logical connectives to build complex propositions

We will discuss the following logical connectives:

- ¬ (not)
- ∧ (conjunction / and)
- ∨ (disjunction / or)
- ⊕ (exclusive or)
- → (implication)
- ↔ (biconditional)

Negation

The negation of a proposition is true iff the proposition is false

What we know

What we want to know

One row for each possible value of "what we know"

p	¬p

The truth table for negation

Negation Examples

Negate the following propositions

- Today is Monday
- 21 * 2 = 42

What is the truth value of the following propositions

- ¬(9 is a prime number)
- ¬(Pittsburgh is in Pennsylvania)

Conjunction

The conjunction of two propositions is true iff both propositions are true

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

The truth table for conjunction

2^2 = 4 rows since we know both p and q!

Disjunction

The disjunction of two propositions is true if *at least one* proposition is true

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	

The truth table for disjunction

Conjunction and disjunction examples

This symbol means "is defined as" or "is equivalent to"

Let:

- $p \equiv x^2 > 0$
- $q \equiv$ A lion weighs less than a mouse
- $r \equiv 10 < 7$
- $s \equiv$ Pittsburgh is located in Pennsylvania

What are the truth values of these expressions:

- $p \wedge q$
- $p \wedge s$
- $p \vee q$
- $q \vee r$

Exclusive or (XOR)

The exclusive or of two propositions is true if *exactly one* proposition is true

p	q	$p \oplus q$
T	T	
T	F	
F	T	
F	F	

The truth table for exclusive or

Note: Exclusive or is typically used to natural language to identify *choices*. For example “You may have a soup or salad with your entree.”

Implication

The implication $p \rightarrow q$ is false if p is true and q is false, and true otherwise

Terminology

- p is called the hypothesis
- q is called the conclusion

p	q	$p \rightarrow q$
T	T	
T	F	
F	T	
F	F	

The truth table for implication

Implication (cont.)

The implication $p \rightarrow q$ can be read in a number of (equivalent) ways:

- If p then q
- p only if q
- p is sufficient for q
- q whenever p

Implication examples

Let:

- $p \equiv$ Jane gets a 100% on her final
- $q \equiv$ Jane gets an A

What are the truth values of these implications:

- $p \rightarrow q$
- $q \rightarrow p$

Other conditional statements

Given an implication p → q:

- q → p is its converse
- ¬q → ¬p is its contrapositive
- ¬p → ¬q is its inverse

Why might this be useful?

Note: An implication and its contrapositive *always* have the same truth value

Biconditional

The biconditional p ↔ q is true if and only if p and q assume the same truth value

p	q	p ↔ q
T	T	
T	F	
F	T	
F	F	

The truth table for the biconditional

Note: The biconditional statement p ↔ q is often read as "p if and only if q" or "p is a necessary and sufficient condition for q."

Truth tables can also be made for more complex expressions

Example: What is the truth table for $(p \wedge q) \rightarrow \neg r$?

Subexpressions of "what we want to know"

What we want to know

p	q	r			

$2^3 = 8$ rows

Like mathematical operators, logical operators are assigned precedence levels

1. Negation
   - $\neg q \vee r$ means $(\neg q) \vee r$, not $\neg(q \vee r)$
2. Conjunction
3. Disjunction
   - $q \wedge r \vee s$ means $(q \wedge r) \vee s$, not $q \wedge (r \vee s)$
4. Implication
   - $q \wedge r \rightarrow s$ means $(q \wedge r) \rightarrow s$, not $q \wedge (r \rightarrow s)$
5. Biconditional

In general, we will try to use parenthesis to disambiguate these types of expressions.

Group Exercises

Problem 1: Show that an implication $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ always have the same value

- **Hint:** Construct two truth tables

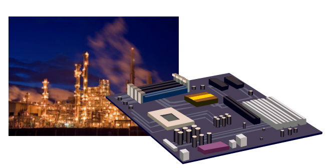
Problem 2: Construct the truth table for the compound proposition $p \wedge (\neg q \vee r) \rightarrow s$

English sentences can often be translated into propositional sentences

But *why* would we do that?!?

Reasoning about law

Philosophy and epistemology



Verifying complex system specifications

Example #1

Example: You can see an R-rated movie only if you are over 17 or you are accompanied by your legal guardian.

Find logical connectives

Let:

Translate fragments

Create logical expression

Example #2

Example: You can have free coffee if you are senior citizen and it is a Tuesday

Let:

Example #3

Example: If you are under 17 and are not accompanied by your legal guardian, then you cannot see the R-rated movie.

Let:

Note: The above translation is the contrapositive of the translation from example 1!

Logic also helps us understand bitwise operations

- Computers represent data as sequences of bits
  - E.g., 0101 1101 1010 1111
- Bitwise logical operations are often used to manipulate these data
- If we treat 1 as true and 0 as false, our logic truth tables tell us how to carry out bitwise logical operations

Bitwise logic examples

$$
\begin{array}{r}
1010\ 1110 \\
\wedge\ 1110\ 1010 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{r}
1010\ 1110 \\
\vee\ 1110\ 1010 \\
\hline
\end{array}
$$

$$
\begin{array}{r}
1010\ 1110 \\
\oplus\ 1110\ 1010 \\
\hline
\end{array}
$$

Group Exercises

Problem 1: Translate the following sentences

- If it is raining then I will either play video games or watch a movie
- You get a free salad only if you order off of the extended menu and it is a Wednesday

Problem 2: Solve the following bitwise problems

$$
\begin{array}{r}
1011\ 1000 \\
\oplus\ 1010\ 0110 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{r}
1011\ 1000 \\
\wedge\ 1010\ 0110 \\
\hline
\end{array}
$$

Final Thoughts

- Propositional logic is a simple logic that allows us to reason about a variety of concepts