# Logistic Regression

Chapter 5



## Classification

- **Learn**: f:X->Y
  - X – features
  - Y – target classes

# Generative vs. Discriminative Models

## Generative

- Learn a model of the joint probability p(d, c)
- Use Bayes' Rule to calculate p(c|d)
- Build a model of each class; given example, return the model most likely to have generated that example
- Examples: Naive Bayes, HMM

## Discriminative

# Naive Bayes Review

- Features = {I hate love this book}
- Training
  - I hate this book
  - Love this book
- What is P(Y|X)?
- Prior p(Y)
- Testing
  - hate book
- Different conditions
  - a = 0 (no smoothing)
  - a = 1 (smoothing)

$$P(Y) = [1/2 \quad 1/2]$$

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$P(X|Y) = \begin{bmatrix} 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$P(Y|X) \propto [1/2 \times 1/4 \times 1/4 \quad 1/2 \times 0 \times 1/3] = [1 \ 0]$$

$$M = \begin{bmatrix} 2 & 2 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 \end{bmatrix}$$

$$P(X|Y) = \begin{bmatrix} 2/9 & 2/9 & 1/9 & 2/9 & 2/9 \\ 1/8 & 1/8 & 2/8 & 2/8 & 2/8 \end{bmatrix}$$

$$P(Y|X) \propto [1/2 \times 2/9 \times 2/9 \quad 1/2 \times 1/8 \times 2/8] = [0.613 \ 0.387]$$

# Generative vs. Discriminative Models

## Generative

- Learn a model of the joint probability p(d, c)
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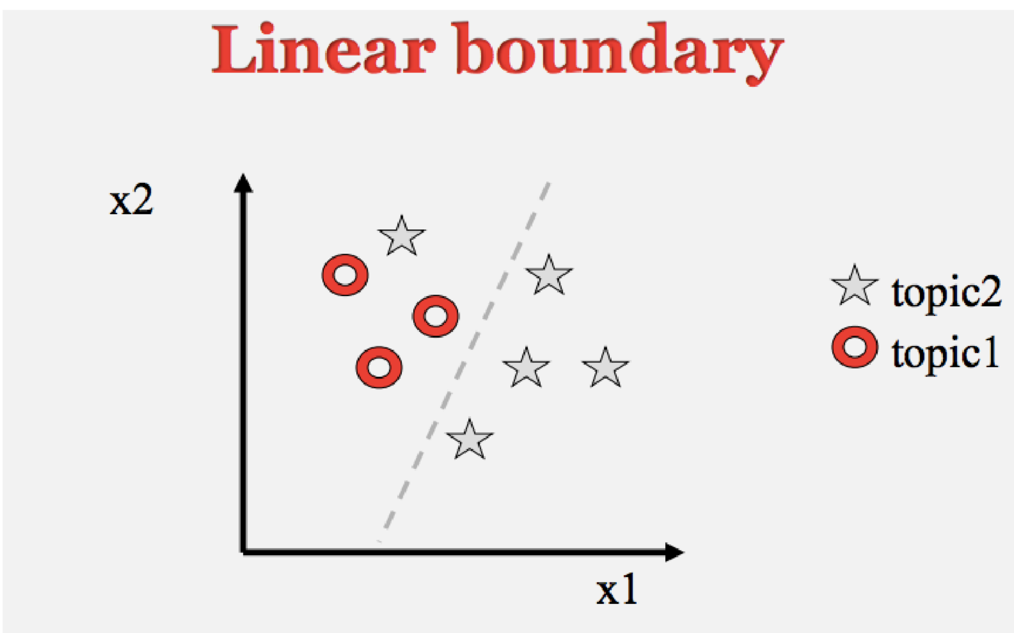
- Examples: Naive Bayes, HMM

## Discriminative

- Model p(c|d) directly
- Class is a function of document vector
- Find the exact function that minimizes classification errors on the training data
- Learn boundaries between classes
- Example: Logistic regression



# Discriminative vs. Generative Classifiers

- Discriminative classifiers are generally more effective, since they directly optimize the classification accuracy. But
  - They are sensitive to the choice of features
    - Plus: easy to incorporate linguistic information
    - Minus: until neural networks, features extracted heuristically
  - Also, overfitting can happen if data is sparse
- Generative classifiers are the “opposite”
  - They directly model text, an unnecessarily harder problem than classification

# Review

- Multiclass NB and Evaluation
- NB tailored to sentiment
- Generative vs discriminative classifiers

# Assumptions of Discriminative Classifiers

- Data examples (documents) are represented as vectors of features (words, phrases, ngrams, etc)
- Looking for a function that maps each vector into a class.
- This function can be found by minimizing the errors on the training data (plus other various criteria)
- Different classifiers vary on what the function looks like, and how they find the function

# Linear Separators

$f(x) = \Theta X + b$

where

$\Theta$ is a vector of weights: $w_1 \ldots w_n$

X is the input vector

b is a constant

Two dimensional space:

$w_1x_1+w_2x_2=b$

In n-dimensional spaces:

$\Theta X=\Sigma_{i=1}^{n}=w_1x_1+w_2x_2+\ldots+w_nx_n$

One can also add $w_0=1$, $x_0=b$ to account for bias

Pass output of f(x) to the sign function, mapping negative values to -1 and positive values to 1

# How to find the weights?

- Logistic regression is one method
- Training using optimization
  - Select values for w
  - Compute f(x)
  - Compare f(x) output to gold labels and compute loss
    - Cross-Entropy (Section 5.3)
  - Adjust w



# What does a logistic regression model look like?

- Given document instance x and sentiment label y
- We can propose various features that we think will tell us whether y is + or -:
  - f1(x): Is the word "excellent" used in x?
  - f2(x): How many adjectives are used in x?
  - f3(x): How many words in x are from the positive list in our sentiment lexicon?
  - ...
- We then need some way to combine these features to help us predict y

# A Feature Representation of the Input

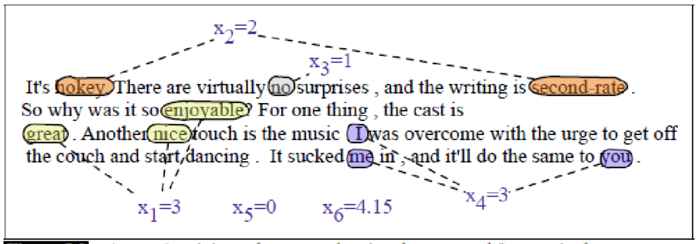



**Figure 5.2** A sample mini test document showing the extracted features in the vector $x$.

| Var | Definition | Value in Fig. 5.2 |
|---|---|---|
| $x_1$ | count(positive lexicon) $\in$ doc) | 3 |
| $x_2$ | count(negative lexicon) $\in$ doc) | 2 |
| $x_3$ | $\begin{cases} 1 & \text{if "no"} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$ | 1 |
| $x_4$ | count(1st and 2nd pronouns $\in$ doc) | 3 |
| $x_5$ | $\begin{cases} 1 & \text{if "!"} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$ | 0 |
| $x_6$ | log(word count of doc) | $\ln(64) = 4.15$ |

But where did the feature representation (and interactions) come from?



# Classification Decision

$$\hat{y} = \begin{cases} 1 & \text{if } P(y=1|x) > 0.5 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} p(+|x) = P(Y=1|x) &= \sigma(w \cdot x + b) \\ &= \sigma([2.5, -5.0, -1.2, 0.5, 2.0, 0.7] \cdot [3, 2, 1, 3, 0, 4.15] + 0.1) \\ &= \sigma(.805) \\ &= 0.69 \qquad (5.6) \\ p(-|x) = P(Y=0|x) &= 1 - \sigma(w \cdot x + b) \\ &= 0.31 \end{aligned}$$

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But where did the weights and bias come from?

# Motivating Logistic Regression, continued

- if f1(x) + f2(x) + … + fn(x) > thresh: return + else return -
  - Problem: not all features are equally important
- if $w_0 + w_1 f_1(x) + \ldots + w_n f_n(x) > 0$: return + else return –
  - Problem: not probabilistic
- Apply sigmoid function

$$f(x) = \frac{1}{1+e^{-x}}$$



# Using a loss function

- Training data
  - $x_1\ x_2\ \ldots .\ x_n$ (input)
  - $y_1 y_2 \ldots .\ y_n$ (labels)
- Algorithm that returns f(x) with predictions ŷ
- Loss function L(ŷ,y)
- Parameters of the learned function (Θ,b) set to minimize L

# Logistic Regression

- An example of a discriminative classifier
- Input:
  - Training example pairs of $(\vec{x}, y)$ where $\vec{x}$ is the feature vector and $y$ is the label
- Goal:
  - Build a model that predicts the probability of the label
- Output:
  - Set of weights $\vec{w}$ that maximizes likelihood of correct labels on training examples

# Logistic Regression

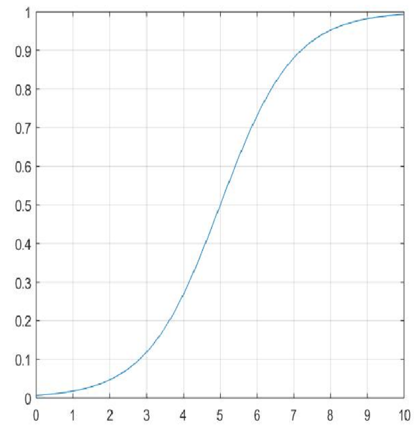
- Similar to Naive Bayes (but discriminative!)
  - Features don't have to be independent
- Examples of features
  - Anything of use
  - Linguistic and non-linguistic
  - Count of "good"
  - Count of "not good"
  - Sentence length

## Classification using LR

- Compute the feature vector *x*
- Multiply with weight vector *w*

$$z = \sum w_i x_i$$

- Compute the logistic sigmoid function




$$f(z) = \frac{1}{1 + e^{-z}}$$

## Examples

- Example 1

  $x = (2,1,1,1)$

  $w = (1,-1,-2,3)$

  $z = 2-1-2+3=2$

  $f(z) = 1/(1+e^{-2})$

- Example 2

  $x = (2,1,0,1)$

  $w = (0,0,-3,0)$

  $z = 0$

  $f(z) = 1/(1+e^{0}) = 1/2$

# Why Sigmoid?
# First, Linear Regression

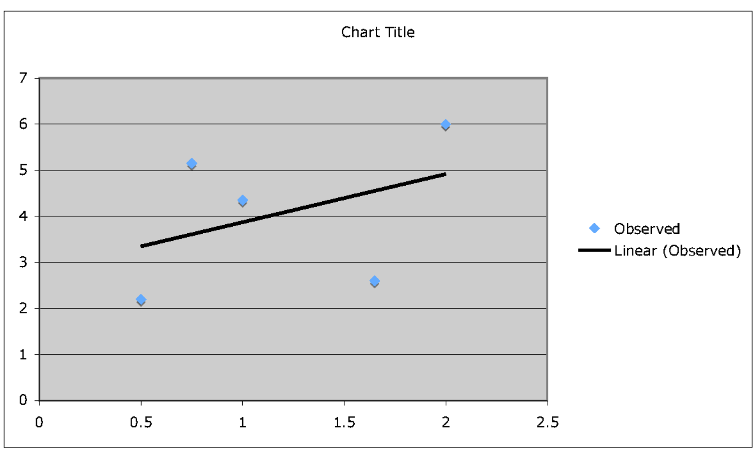
- Regression used to fit a linear model to data where the dependent variable is continuous:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_n X_n + \varepsilon$$

- Given a set of points (Xi,Yi), we wish to find a linear function (or line in 2 dimensions) that "goes through" these points.
- In general, the points are not exactly aligned:
  - Find line that best fits the points

# Error

- Error:
  - Observed value - Predicted value



# Logistic Regression

- Regression used to fit a curve to data in which the dependent variable is binary, or dichotomous
- Example application: Medicine
  - We might want to predict response to treatment, where we might code survivors as 1 and those who don't survive as 0

# Example

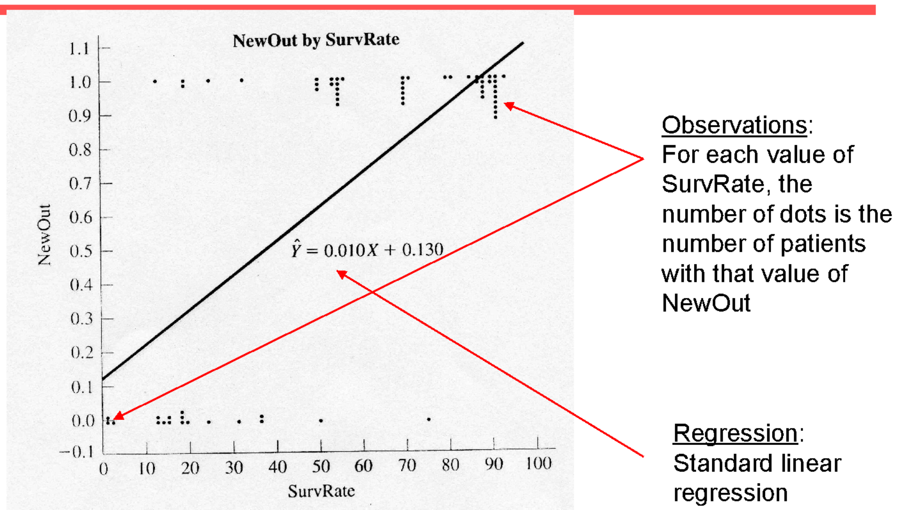



FIGURE 15.7 *Outcome as a function of SurvRate*

Problem: extending the regression line a few units left or right along the X axis produces predicted probabilities that fall outside of [0,1]

# A Better Solution

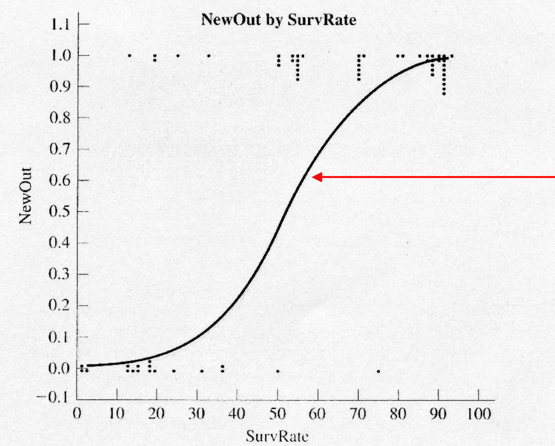
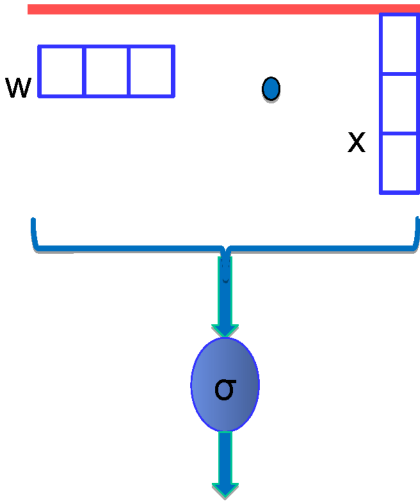



FIGURE 15.8 *More appropriate regression line for predicting outcome*

Regression Curve: Sigmoid function!

(bounded by asymptotes $y$=0 and $y$=1)

# Logistic Regression




$$P(y = C \mid x)$$

## Constructing a Learning Algorithm

- The conditional data likelihood is the probability of the observed Y values in the training data, conditioned on their corresponding X values. We choose parameters w that satisfy

$$\mathbf{w} = \arg\max_{\mathbf{w}} \prod_{l} P(y^l \mid \mathbf{x}^l, \mathbf{w})$$

- where $\mathbf{w} = \langle w_0, w_1, \ldots, w_n \rangle$ is the vector of parameters to be estimated, $y^l$ denotes the observed value of Y in the $l$ th training example, and $\mathbf{x}^l$ denotes the observed value of $\mathbf{X}$ in the $l$ th training example



## Constructing a Learning Algorithm

- Equivalently, we can work with the log of the conditional likelihood:

$$\mathbf{w} = \arg\max_{\mathbf{w}} \sum_{l} \ln P(y^l \mid \mathbf{x}^l, \mathbf{w})$$

- This conditional data log likelihood, which we will denote l(W) can be written as

$$l(\mathbf{w}) = \sum_{l} y^l \ln P(y^l = 1 \mid \mathbf{x}^l, \mathbf{w}) + (1 - y^l) \ln P(y^l = 0 \mid \mathbf{x}^l, \mathbf{w})$$

- Note here we are utilizing the fact that Y can take only values 0 or 1, so only one of the two terms in the expression will be non-zero for any given $y^l$

# Fitting LR by Gradient Descent

- Unfortunately, there is no closed form solution to maximizing l(**w**) with respect to **w**. Therefore, one common approach is to use gradient descent
  - Beginning with initial weights of zero, we repeatedly update the weights
  - Details optional, see text, but should understand following concepts
    - Loss function
    - Gradient descent
      - Gradient, learning rate, mini-batch training
    - Regularization
      - Overfitting



# Gradient Descent

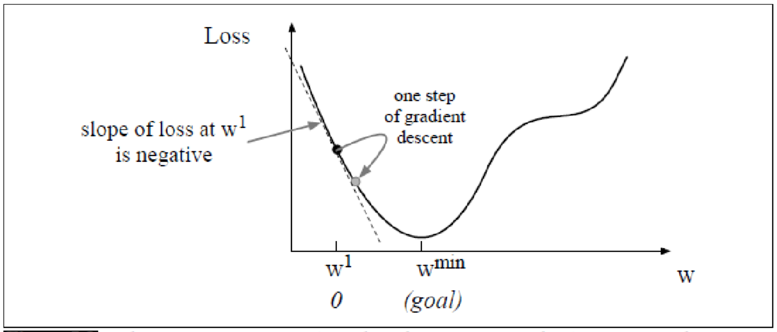



**Figure 5.3** The first step in iteratively finding the minimum of this loss function, by moving $w$ in the reverse direction from the slope of the function. Since the slope is negative, we need to move $w$ in a positive direction, to the right. Here superscripts are used for learning steps, so $w^1$ means the initial value of $w$ (which is 0), $w^2$ at the second step, and so on.

Learning Rate:
- the magnitude of the amount to move is the slope (more generally, the gradient) weighted by the learning rate
- if too high, overshoot minimum
- if too low, take too long to learn
- common to begin high, then decrease

## Some Practical Issues

- Feature representation
  - want all features to have similar value ranges
  - too many features? feature selection
- Efficiency
  - Stochastic Gradient Descent / Batching
- Over-fitting
  - Regularization
- Classifying more than two categories



## Mini-batch training

- Stochastic gradient descent chooses a random example at a time
- To make movements less choppy, compute gradient over batches of training instances from training set of size m
  - If batch size is m, batch training
  - If batch size is 1, stochastic gradient descent
  - Otherwise, mini batch training (for efficiency)

## Regularization

- Weight training can yield models that don't generalize well to test data (i.e., that overfit to training data)
- To avoid overfitting, a regularization term (various options) is used to penalize large weights
  - L2 quadratic function of the weight values
  - L1 linear function of the weight values



## Multinomial Logistic Regression

- More than two classes
- AKA softmax regression, maxent classifier
  - Instead of sigmoid, use "softmax function"
  - Instead of having just one set of weights and one set of features, different set of weights and feature vectors for each class label
  - Loss function changes too

## Summary of Logistic Regression

- Learns the Conditional Probability Distribution P(y|x)
- Local Search.
  - Begins with initial weight vector.
  - Modifies it iteratively to maximize an objective function.
  - The objective function is the conditional log likelihood of the data – so the algorithm seeks the probability distribution P(y|x) that is most likely given the data.



## Two Phases

- Training
  - we train the system (specifically the weights w and b), e.g., using stochastic gradient descent and the cross-entropy loss
- Test
  - Given a test example x we compute p(y | x) and return the higher probability label y = 1 or y = 0

## Final Comments

- In general, NB and LR make different assumptions
  - NB: Features independent given class -> assumption on P(X|Y)
  - LR: Functional form of P(Y|X), no assumption on P(X|Y)
- LR is optimized
  - no closed-form solution
- LR is interpretable



## Summary

- Logistic regression is a supervised machine learning classifier (discriminative)
- Use: LR extracts real-valued features from the input, multiplies each by a weight, sums them, and passes the sum through a sigmoid function to generate a probability. A threshold is used to make a decision
- Learning: The weights (vector w and bias b) are learned from a labeled training set via a loss function that must be minimized, e.g., by using (iterative) gradient descent to find the optimal weights, and regularization to avoid overfitting