W[1]-hardness

Dániel Marx

Recent Advances in Parameterized Complexity
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Lower bounds

So far we have seen positive results: basic algorithmic techniques for fixed-parameter tractability.

What kind of negative results we have?

- Can we show that a problem (e.g., CLIQUE) is **not** FPT?
- Can we show that a problem (e.g., VERTEX COVER) has **no** algorithm with running time, say, $2^{o(k)} \cdot n^{O(1)}$?

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This would require showing that $\mathrm{P} \neq \mathrm{NP}$: if $\mathrm{P} = \mathrm{NP}$, then, e.g., k-CLIQUE is polynomial-time solvable, hence FPT.

Can we give some evidence for negative results?

Goals of this talk

Two goals:

1. Explain the theory behind parameterized intractability.
2. Show examples of parameterized reductions.

Classical complexity

Nondeterministic Turing Machine (NTM): single tape, finite alphabet, finite state, head can move left/right only one cell. In each step, the machine can branch into an arbitrary number of directions. Run is successful if at least one branch is successful.

NP: The class of all languages that can be recognized by a polynomial-time NTM.

Polynomial-time reduction from problem P to problem Q: a function ϕ with the following properties:

- $\phi(x)$ is a yes-instance of $Q \iff x$ is a yes-instance of P,
- $\phi(x)$ can be computed in time $|x|^{O(1)}$.

Definition: Problem Q is NP-hard if any problem in NP can be reduced to Q.

If an NP-hard problem can be solved in polynomial time, then every problem in NP can be solved in polynomial time (i.e., $\mathrm{P} = \mathrm{NP}$).

Parameterized complexity

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- An appropriate notion of reduction.
- An appropriate hypothesis.

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Example: Graph G has an independent set k if and only if it has a vertex cover of size $n - k$.

$\Rightarrow$ Transforming an INDEPENDENT SET instance (G, k) into a VERTEX COVER instance $(G, n - k)$ is a correct polynomial-time reduction.

However, VERTEX COVER is FPT, but INDEPENDENT SET is not known to be FPT.

Parameterized reduction

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Parameterized reduction from problem P to problem Q: a function ϕ with the following properties:

- $\phi(x)$ is a yes-instance of $Q \iff x$ is a yes-instance of P,
- $\phi(x)$ can be computed in time $f(k) \cdot |x|^{O(1)}$, where k is the parameter of x,
- If k is the parameter of x and k' is the parameter of $\phi(x)$, then $k' \leq g(k)$ for some function g.

Fact: If there is a parameterized reduction from problem P to problem Q and Q is FPT, then P is also FPT.

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Non-example: Transforming an Independent Set instance (G, k) into a Vertex Cover instance $(G, n-k)$ is **not** a parameterized reduction.

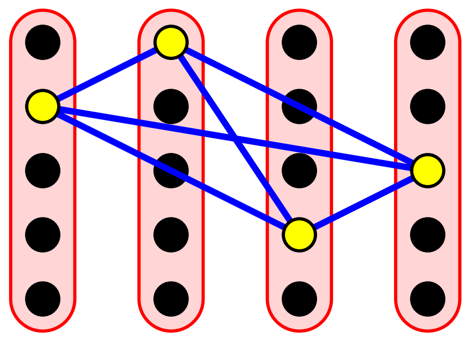
Example: Transforming an Independent Set instance (G, k) into a Clique instance $(\overline{G}, k)$ **is** a parameterized reduction.

Multicolored Clique

A useful variant of Clique:

Multicolored Clique: The vertices of the input graph G are colored with k colors and we have to find a clique containing one vertex from each color.

(or Partitioned Clique)



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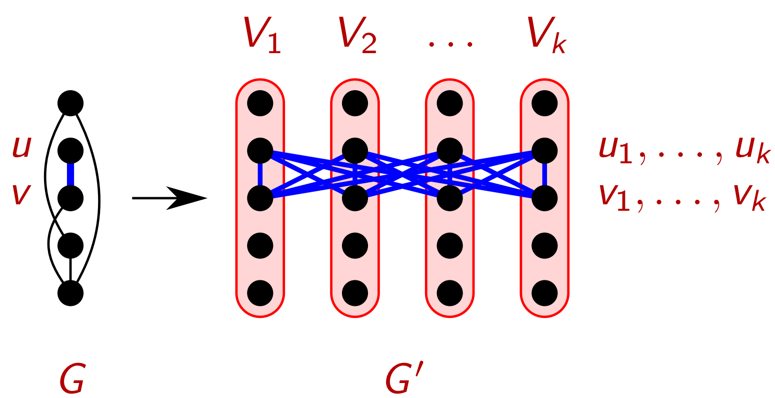
Create G' by replacing each vertex v with k vertices, one in each color class. If u and v are adjacent in the original graph, connect all copies of u with all copies of v.



k-clique in $G \iff$ multicolored k-clique in G'.

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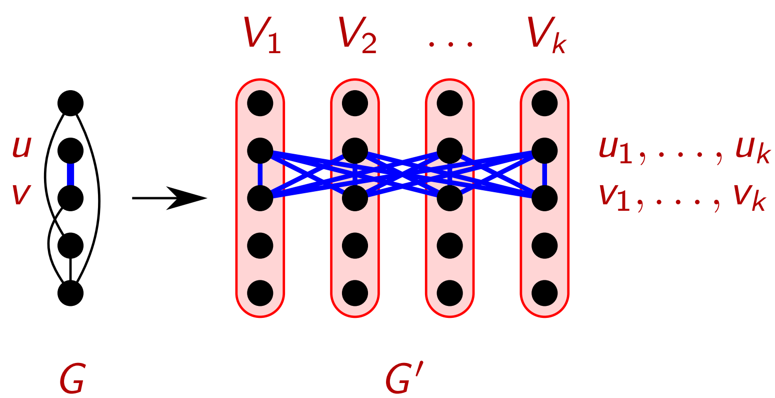


k-clique in G $\iff$ multicolored k-clique in G'.

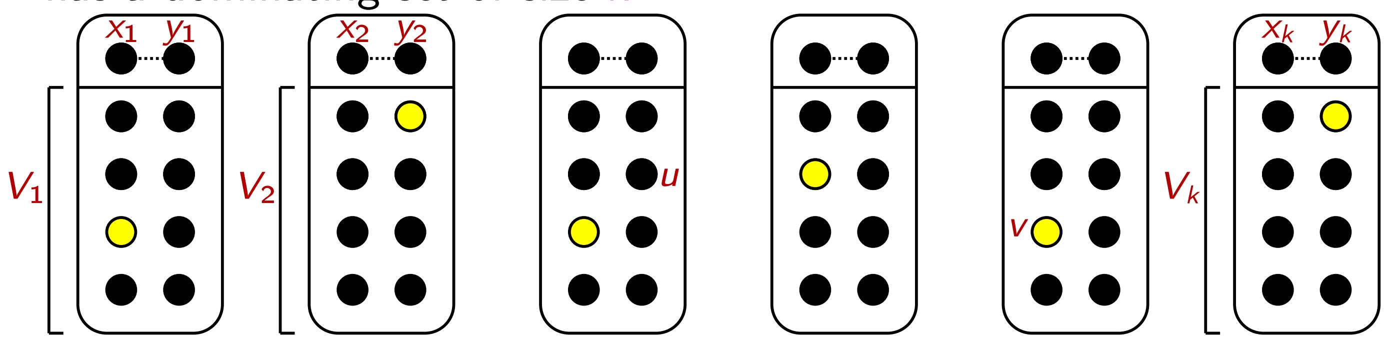
Similarly: reduction to Multicolored Independent Set.

Dominating Set

Theorem

There is a parameterized reduction from Multicolored Independent Set to Dominating Set.

Proof: Let G be a graph with color classes $V_1, \dots, V_k$. We construct a graph H such that G has a multicolored k-clique iff H has a dominating set of size k.



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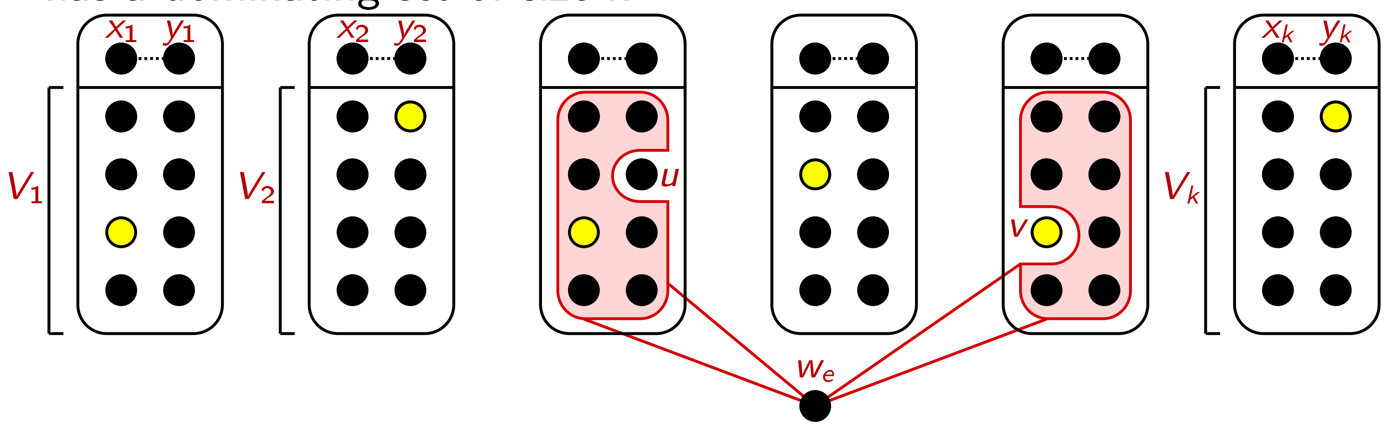

- The dominating set has to contain one vertex from each of the k cliques $V_1, \ldots, V_k$ to dominate every x_i and y_i.
- For every edge $e = uv$, an additional vertex w_e ensures that these selections describe an independent set.

Variants of DOMINATING SET

- DOMINATING SET: Given a graph, find k vertices that dominate every vertex.
- RED-BLUE DOMINATING SET: Given a bipartite graph, find k vertices on the red side that dominate the blue side.
- SET COVER: Given a set system, find k sets whose union covers the universe.
- HITTING SET: Given a set system, find k elements that intersect every set in the system.

All of these problems are equivalent under parameterized reductions, hence at least as hard as CLIQUE.

Basic hypotheses

It seems that parameterized complexity theory cannot be built on assuming $P \neq NP$ – we have to assume something stronger.

Let us choose a basic hypothesis:

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k-CLIQUE cannot be solved in time $f(k) \cdot n^{O(1)}$.

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n-variable 3SAT cannot be solved in time $2^{o(n)}$.

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Summary

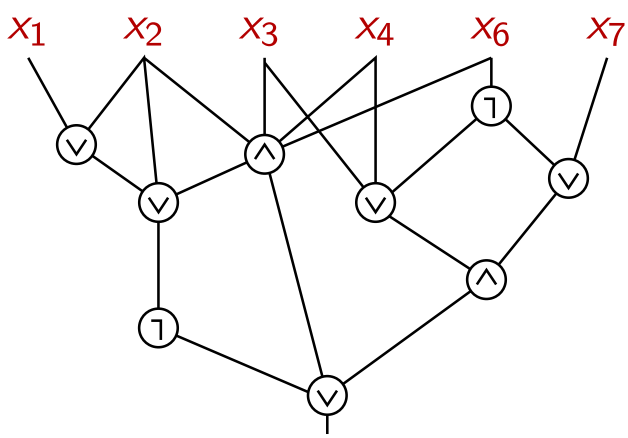
- INDEPENDENT SET and k-STEP HALTING PROBLEM can be reduced to each other $\Rightarrow$ Engineers' Hypothesis and Theorists' Hypothesis are equivalent!
- INDEPENDENT SET and k-STEP HALTING PROBLEM can be reduced to DOMINATING SET.

Summary

- Independent Set and k-Step Halting Problem can be reduced to each other ⇒ Engineers' Hypothesis and Theorists' Hypothesis are equivalent!
- Independent Set and k-Step Halting Problem can be reduced to Dominating Set.
- Is there a parameterized reduction from Dominating Set to Independent Set?
- Probably not. Unlike in NP-completeness, where most problems are equivalent, here we have a hierarchy of hard problems.
    - Independent Set is W[1]-complete.
    - Dominating Set is W[2]-complete.
- Does not matter if we only care about whether a problem is FPT or not!

Boolean circuit

A **Boolean circuit** consists of input gates, negation gates, AND gates, OR gates, and a single output gate.



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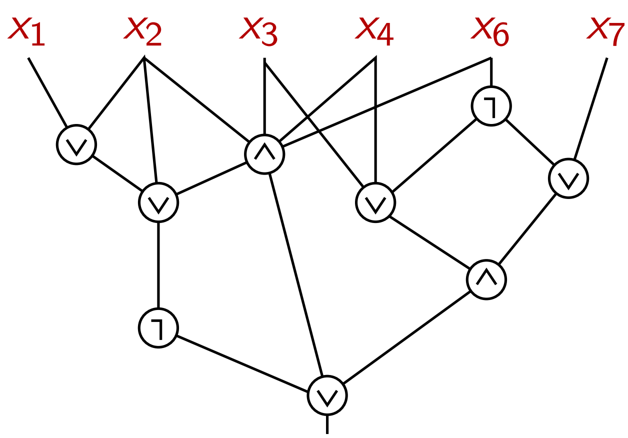

CIRCUIT SATISFIABILITY: Given a Boolean circuit C, decide if there is an assignment on the inputs of C making the output true.

Weight of an assignment: number of true values.

WEIGHTED CIRCUIT SATISFIABILITY: Given a Boolean circuit C and an integer k, decide if there is an assignment of weight k making the output true.