Lecture #27: Relations and Representations
**Definition:** Let $A$ and $B$ be two sets. A **binary relation** from $A$ to $B$ is a subset of $A \times B$.

In other words, a binary relation $R$ is a set of ordered pairs $(a_i, b_i)$ where $a_i \in A$ and $b_i \in B$.

**Notation:** We say that
- $a R b$ if $(a,b) \in R$
- $a \not R b$ if $(a,b) \notin R$
Example: Course Enrollments

Let’s say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation R that represents the relationship between people and classes.

Solution:

- Let the set P denote people, so P = {Alice, Bob, Charlie}
- Let the set C denote classes, so C = {CS 441, Math 336, Art 212, Business 444}
- By definition R ⊆ P × C
- From the above statement, we know that
  - (Alice, CS 441) ∈ R
  - (Bob, CS 441) ∈ R
  - (Alice, Math 336) ∈ R
  - (Charlie, Art 212) ∈ R
  - (Charlie, Business 444) ∈ R
- So, R = {(Alice, CS 441), (Bob, CS 441), (Alice, Math 336), (Charlie, Art 212), (Charlie, Business 444)}
A relation can also be represented as a graph

Let’s say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation $R$ that represents the relationship between people and classes.

$\{(\text{Alice}, \text{CS 441}) \in R\}$

Elements of $P$ (i.e., people)

Elements of $C$ (i.e., classes)
A relation can also be represented as a table

Let’s say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation \( R \) that represents the relationship between people and classes.

<table>
<thead>
<tr>
<th>Name of the relation</th>
<th>Elements of C (i.e., courses)</th>
<th>Elements of P (i.e., people)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>Art 212</td>
<td>Business 444</td>
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<tr>
<td>Alice</td>
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<td>CS 441</td>
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<tr>
<td>Bob</td>
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<td>Math 336</td>
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<tr>
<td>Charlie</td>
<td>X</td>
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</tbody>
</table>

\( (Bob, CS 441) \in R \)
Wait, doesn’t this mean that relations are the same as functions?

Not quite... Recall the following definition from Lecture #9.

Definition: Let A and B be nonempty sets. A function, f, is an assignment of exactly one element of set B to each element of set A.

This would mean that, e.g., a person only be enrolled in one course!

Reconciling this with our definition of a relation, we see that

1. Every function is also a relation
2. Not every relation is a function

Let’s see some quick examples...
1. Consider $f : S \rightarrow G$
   - Clearly a function
   - Can also be represented as the relation $R = \{(\text{Anna, C}), (\text{Brian, A}), (\text{Christine, A})\}$

1. Consider the set $R = \{(A, 1), (A, 2)\}$
   - Clearly a relation
   - Cannot be represented as a function!
We can also define binary relations on a single set

**Definition:** A relation on the set $A$ is a relation from $A$ to $A$. That is, a relation on the set $A$ is a subset of $A \times A$.

**Example:** Let $A$ be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

**Solution:**

- 1 divides everything
- 2 divides itself and 4
- 3 divides itself
- 4 divides itself

So, $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
Example: Let $A$ be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?
**Question:** Which of the following relations contain each of the pairs (1,1), (1,2), (2,1), (1,-1), and (2,2)?

- $R_1 = \{(a,b) \mid a \leq b\}$
- $R_2 = \{(a,b) \mid a > b\}$
- $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a,b) \mid a = b\}$
- $R_5 = \{(a,b) \mid a = b + 1\}$
- $R_6 = \{(a,b) \mid a + b \leq 3\}$

**Answer:**

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<tr>
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<th>(1,1)</th>
<th>(1,2)</th>
<th>(2,1)</th>
<th>(1,-1)</th>
<th>(2,2)</th>
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These are all relations on an infinite set!
**Definition:** A relation $R$ on a set $A$ is reflexive if $(a,a) \in R$ for every $a \in A$.

**Note:** Our “divides” relation on the set $A = \{1,2,3,4\}$ is reflexive.
Properties of Relations

**Definition:** A relation R on a set A is **symmetric** if \((b,a) \in R\) whenever \((a,b) \in R\) for every \(a,b \in A\). If R is a relation in which \((a,b) \in R\) and \((b,a) \in R\) implies that \(a=b\), we say that R is **antisymmetric**.

**Mathematically:**

- Symmetric: \(\forall a \forall b ((a,b) \in R \rightarrow (b,a) \in R)\)
- Antisymmetric: \(\forall a \forall b (((a,b) \in R \land (b,a) \in R) \rightarrow (a = b))\)

**Examples:**

- Symmetric: \(R = \{(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)\}\)
- Antisymmetric: \(R = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)\}\)
### Symmetric and Antisymmetric Relations

#### Symmetric relation
- Diagonal axis of symmetry
- Not all elements on the axis of symmetry need to be included in the relation

\[ R = \{(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)\} \]

#### Antisymmetric relation
- No axis of symmetry
- Only symmetry occurs on diagonal
- Not all elements on the diagonal need to be included in the relation

\[ R = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)\} \]

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Properties of Relations

**Definition:** A relation R on a set A is **transitive** if, whenever \((a,b) \in R\) and \((b,c) \in R\), then \((a,c) \in R\) for every \(a,b,c \in A\).

**Note:** Our “divides” relation on the set \(A = \{1,2,3,4\}\) is transitive.

Recall that: \(a|b \land b|c \rightarrow a|c\)

More common transitive relations include equality and comparison operators like \(<, >, \leq,\) and \(\geq\).
Examples, redux

**Question:** Which of the following relations are reflexive, symmetric, antisymmetric, and/or transitive?

- $R_1 = \{(a,b) \mid a \leq b\}$
- $R_2 = \{(a,b) \mid a > b\}$
- $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a,b) \mid a = b\}$
- $R_5 = \{(a,b) \mid a = b + 1\}$
- $R_6 = \{(a,b) \mid a + b \leq 3\}$

**Answer:**

<table>
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<tr>
<th></th>
<th>Reflexive</th>
<th>Symmetric</th>
<th>Antisymmetric</th>
<th>Transitive</th>
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Example: Let $R$ be the relation that pairs students with courses that they have taken. Let $S$ be the relation that pairs students with courses that they need to graduate. What do the relations $R \cup S$, $R \cap S$, and $S - R$ represent?

Solution:

- $R \cup S =$ All pairs $(a,b)$ where
  - student $a$ has taken course $b$ OR
  - student $a$ needs to take course $b$ to graduate

- $R \cap S =$ All pairs $(a,b)$ where
  - Student $a$ has taken course $b$ AND
  - Student $a$ needs course $b$ to graduate

- $S - R =$ All pairs $(a,b)$ where
  - Student $a$ needs to take course $b$ to graduate BUT
  - Student $a$ has not yet taken course $b$
**Definition:** Let \( R \) be a relation from the set \( A \) to the set \( B \), and \( S \) be a relation from the set \( B \) to the set \( C \). The **composite** of \( R \) and \( S \) is the relation of ordered pairs \( (a, c) \), where \( a \in A \) and \( c \in C \) for which there exists an element \( b \in B \) such that \( (a, b) \in R \) and \( (b, c) \in S \). We denote the composite of \( R \) and \( S \) by \( R \circ S \).

**Example:** What is the composite relation of \( R \) and \( S \)?

\[
\begin{align*}
R: \{1,2,3\} &\to \{1,2,3,4\} \\
S: \{1,2,3,4\} &\to \{0,1,2\}
\end{align*}
\]

\[
\begin{align*}
R &= \{(1,1),(1,4),(2,3),(3,1),(3,4)\} \\
S &= \{(1,0),(2,0),(3,1),(3,2),(4,1)\}
\end{align*}
\]

So: \( R \circ S = \{(1,0), (3,0), (1,1), (3,1), (2,1), (2,2)\} \)
Problem 1: List the ordered pairs of the relation $R$ from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ iff $a + b = 4$.

Problem 2: Draw the graph and table representations of the above relation.

Problem 3: Is the above relation reflexive, symmetric, antisymmetric, and/or transitive?
Final Thoughts

Relations allow us to represent and reason about the relationships between sets.

Relations are more general than functions.

Relations are use all over...
- Mathematical operators
- Bindings between sets of objects
- Etc.

Next time: n-ary relations