Lecture #25: Relations and Representations

Based on materials developed by Dr. Adam Lee
Binary relations establish a relationship between elements of two sets

**Definition:** Let A and B be two sets. A binary relation from A to B is a subset of $A \times B$.

In other words, a binary relation $R$ is a set of ordered pairs $(a_i, b_i)$ where $a_i \in A$ and $b_i \in B$.

**Notation:** We say that
- $a R b$ if $(a, b) \in R$
- $a \notin R b$ if $(a, b) \notin R$
Let’s say that Ankita and Braden are taking CS 441. Ankita is also taking Math 336. Furthermore, Carys is taking Art 212 and Business 444. Define a relation $R$ that represents the relationship between people and classes.

**Solution:**

- Let the set $P$ denote people, so $P = \{\text{Ankita, Braden, Carys}\}$
- Let the set $C$ denote classes, so $C = \{\text{CS 441, Math 336, Art 212, Business 444}\}$
- By definition, $R \subseteq P \times C$
- From the above statement, we know that
  - $(\text{Ankita, CS 441}) \in R$
  - $(\text{Braden, CS 441}) \in R$
  - $(\text{Ankita, Math 336}) \in R$
  - $(\text{Carys, Art 212}) \in R$
  - $(\text{Carys, Business 444}) \in R$
- So, $R = \{(\text{Ankita, CS 441}), (\text{Braden, CS 441}), (\text{Ankita, Math 336}), (\text{Carys, Art 212}), (\text{Carys, Business 444})\}$
A relation can be represented as a directed graph

Let’s say that Ankita and Braden are taking CS 441. Ankita is also taking Math 336. Furthermore, Carys is taking Art 212 and Business 444. Define a relation $R$ that represents the relationship between people and classes.

$R = \{(\text{Ankita}, \text{CS 441}), (\text{Braden}, \text{Art 212}), (\text{Carys}, \text{CS 441}), (\text{Carys}, \text{Business 444}), (\text{Ankita}, \text{Math 336})\}$

Elements of $P$ (i.e., people):
- Ankita
- Braden
- Carys

Elements of $C$ (i.e., classes):
- Art 212
- Business 444
- CS 441
- Math 336
A relation can be represented as a table, or 0,1-matrix.

Let’s say that Ankita and Braden are taking CS 441. Ankita is also taking Math 336. Furthermore, Carys is taking Art 212 and Business 444. Define a relation $R$ that represents the relationship between people and classes.

<table>
<thead>
<tr>
<th>Name of the relation</th>
<th>Elements of $C$ (i.e., courses)</th>
<th>Elements of $P$ (i.e., people)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Braden, CS 441) ∈ $R$</td>
<td>Ankita, Business 444, CS 441, Math 336</td>
<td>Ankita, Braden, Carys</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ankita</th>
<th>Business 444</th>
<th>CS 441</th>
<th>Math 336</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Braden</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Carys</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>
Wait, doesn’t this mean that relations are the same as functions?

Not quite... Recall the following definition from Lecture #9.

Definition: Let A and B be nonempty sets. A function, f, is an assignment of exactly one element of set B to each element of set A.

This would mean that, e.g., a person only be enrolled in one course!

Reconciling this with our definition of a relation, we see that

1. Every function also defines a relation
2. Not every relation is a valid function

Let’s see some quick examples...
1. Consider $f : S \rightarrow G$
   - Clearly a function
   - Can also be represented as the relation $R = \{(\text{Anna}, \text{C}), (\text{Brian}, \text{A}), (\text{Christine}, \text{A})\}$

1. Consider the set $R = \{(A, 1), (A, 2)\}$
   - Clearly a relation
   - Cannot be represented as a function!
We can also define binary relations on a single set

**Definition:** A relation on the set $A$ is a relation from $A$ to $A$. That is, a relation on the set $A$ is a subset of $A \times A$.

**Example:** Let $A$ be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

**Solution:**
- $1$ divides everything
- $2$ divides itself and $4$
- $3$ divides itself
- $4$ divides itself

So, $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
Representing the last example as a graph...

**Example:** Let A be the set \(\{1, 2, 3, 4\}\). Which ordered pairs are in the relation \(R = \{(a, b) \mid a \text{ divides } b\}\)?
**Question:** Which of the following relations contain each of the pairs (1,1), (1,2), (2,1), (1,-1), and (2,2)?

- $R_1 = \{(a,b) \mid a \leq b\}$
- $R_2 = \{(a,b) \mid a > b\}$
- $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a,b) \mid a = b\}$
- $R_5 = \{(a,b) \mid a = b + 1\}$
- $R_6 = \{(a,b) \mid a + b \leq 3\}$

**Answer:**

<table>
<thead>
<tr>
<th></th>
<th>(1,1)</th>
<th>(1,2)</th>
<th>(2,1)</th>
<th>(1,-1)</th>
<th>(2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_6$</td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

These are all relations on an infinite set!
**Definition:** A relation $R$ on a set $A$ is reflexive if $(a, a) \in R$ for every $a \in A$.

**Note:** Our “divides” relation on the set $A = \{1, 2, 3, 4\}$ is reflexive.

Every $a \in A$ divides itself!
**Properties of Relations**

**Definition:** A relation $R$ on a set $A$ is **symmetric** if $(b,a) \in R$ whenever $(a,b) \in R$ for every $a,b \in A$. If $R$ is a relation in which $(a,b) \in R$ and $(b,a) \in R$ implies that $a=b$, we say that $R$ is **antisymmetric**.

**Mathematically:**
- Symmetric: $\forall a \forall b ((a,b) \in R \rightarrow (b,a) \in R)$
- Antisymmetric: $\forall a \forall b (((a,b) \in R \land (b,a) \in R) \rightarrow (a = b))$

**Examples:**
- Symmetric: $R = \{(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)\}$
- Antisymmetric: $R = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)\}$
Symmetric and Antisymmetric Relations

**Symmetric relation**
- Diagonal axis of symmetry
- Not all elements on the axis of symmetry need to be included in the relation

**Antisymmetric relation**
- No axis of symmetry
- Only symmetry occurs on diagonal
- Not all elements on the diagonal need to be included in the relation

R = {(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)}

R = {(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)}
**Definition:** A relation $R$ on a set $A$ is **transitive** if, whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ for every $a,b,c \in A$.

**Note:** Our “divides” relation on the set $A = \{1,2,3,4\}$ is transitive.

Recall that: $a | b \land b | c \rightarrow a | c$

More common transitive relations include equality and comparison operators like $<, >, \leq, \text{ and } \geq$. 
Examples, redux

**Question:** Which of the following relations are reflexive, symmetric, antisymmetric, and/or transitive?

- $R_1 = \{(a,b) \mid a \leq b\}$
- $R_2 = \{(a,b) \mid a > b\}$
- $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a,b) \mid a = b\}$
- $R_5 = \{(a,b) \mid a = b + 1\}$
- $R_6 = \{(a,b) \mid a + b \leq 3\}$

**Answer:**

<table>
<thead>
<tr>
<th></th>
<th>Reflexive</th>
<th>Symmetric</th>
<th>Antisymmetric</th>
<th>Transitive</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$R_2$</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$R_3$</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>$R_4$</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$R_5$</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$R_6$</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>
Relations can be combined using set operations

**Example:** Let R be the relation that pairs students with courses that they have taken. Let S be the relation that pairs students with courses that they need to graduate. What do the relations $R \cup S$, $R \cap S$, and $S - R$ represent?

**Solution:**

- $R \cup S =$ All pairs $(a,b)$ where
  - student $a$ has taken course $b$ OR
  - student $a$ needs to take course $b$ to graduate

- $R \cap S =$ All pairs $(a,b)$ where
  - Student $a$ has taken course $b$ AND
  - Student $a$ needs course $b$ to graduate

- $S - R =$ All pairs $(a,b)$ where
  - Student $a$ needs to take course $b$ to graduate BUT
  - Student $a$ has not yet taken course $b$
**Definition:** Let \( R \) be a relation from the set \( A \) to the set \( B \), and \( S \) be a relation from the set \( B \) to the set \( C \). The composite of \( R \) and \( S \) is the relation of ordered pairs \((a, c)\), where \( a \in A \) and \( c \in C \) for which there exists an element \( b \in B \) such that \((a, b) \in R\) and \((b, c) \in S\). We denote the composite of \( R \) and \( S \) by \( S \circ R \).

**Example:** What is the composite relation of \( R \) and \( S \)?

\[
R: \{1,2,3\} \rightarrow \{1,2,3,4\} \\
\quad \text{● } R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}
\]

\[
S: \{1,2,3,4\} \rightarrow \{0,1,2\} \\
\quad \text{● } S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}
\]

So: \( S \circ R = \{(1,0), (3,0), (1,1), (3,1), (2,1), (2,2)\} \)

We’ll use \( R^2 \) for \( R \circ R \), \( R^3 \) for \( R \circ R \circ R \), ..., and \( R^* \) for the combination of all arbitrary numbers of compositions of \( R \).
In-class exercises

Problem 1: List the ordered pairs of the relation $R$ on $A = \{0,1,2,3,4\}$ where $(a,b) \in R$ iff $a + b = 4$.

Problem 2: Draw the graph and table representations of the above relation.

Problem 3: Is the above relation reflexive, symmetric, antisymmetric, and/or transitive?
Final Thoughts

Relations allow us to represent and reason about the relationships between sets.

Relations are more general than functions.

Relations are used all over...
- Mathematical operators
- Bindings between sets of objects
- Etc.

Next time: n-ary relations