Discrete Structures for Computer Science

William Garrison

bill@cs.pitt.edu

6311 Sennott Square

Lecture #23: Solving congruences

University of Pittsburgh

Today's Topics

Arithmetic modulo n (reminder)

Solving linear congruences

- Modular inverses
- Extended Euclidean algorithm and Bézout numbers

Solving systems of congruences

- Chinese remainder theorem

Primitive roots and discrete log

Defining arithmetic restricted to remainders when dividing by m

$\mathbf{Z}_m$ denotes the set of nonnegative integers less than m

- i.e., the remainders when dividing by m

Recall that **mod** "preserves" addition and multiplication

- $(a+b) \textbf{ mod } m = \big((a \textbf{ mod } m) + (b \textbf{ mod } m)\big) \textbf{ mod } m$
- $(ab) \textbf{ mod } m = \big((a \textbf{ mod } m)(b \textbf{ mod } m)\big) \textbf{ mod } m$

Thus, we can define versions of addition and multiplication that are restricted to this set

- $a +_m b = (a+b) \textbf{ mod } m$
- $a \cdot_m b = (a \cdot b) \textbf{ mod } m$
- These operations form arithmetic modulo m

Modular arithmetic behaves similarly to standard arithmetic: Recap properties from § 4.1

Solving congruences via inverses

Consider the equation $a + 8 \equiv 2 \pmod{11}$

- In standard arithmetic, we'd subtract 8 from both sides
  - i.e., utilize the additive inverse

In modular arithmetic, additive inverses are easy to compute!

- $-8 \equiv 3 \pmod{11}$
- Thus, we can add 3 to both sides:
  - $a + 8 + 3 \equiv 2 + 3 \pmod{11}$
  - $a + 11 \equiv 5 \pmod{11}$
  - $a \equiv 5 \pmod{11}$
- Note that adding any multiple of m preserves the value (mod m)

Unfortunately, multiplicative inverses are not as simple

We cannot easily "divide by a" mod n

- What is the equivalent of 1/a mod n?

Linear congruences are of the form $ax \equiv b \pmod{m}$

- Given values for a and b, how do we solve for x?
- We need a value, say $\bar{a}$, where $a\bar{a} \equiv 1 \pmod{m}$
- If we had this, we could multiply on both sides, then simplify!

Good news: Bézout's theorem says that there exist integers s and t such that gcd(a, m) = sa + tm

- Assume a and m are coprime: How does this help us?

Extended Euclidean Algorithm

The extended Euclidean algorithm computes the GCD of a and b, **and** computes the Bézout numbers s and t which satisfy the Bézout identity:

$$\gcd(a, b) = sa + tb$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78					
*Let this represent “a **div** b”*					*“a **mod** b”*		
3							
4							
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1				
2							
3							
4							
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2							
3							
4							
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2							
3							
4							
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21					
3							
4							
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3				
3							
4							
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3							
4							
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15					
4							
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1				
4							
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1	6			
4							
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1	6			
4	15	6					
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1	6			
4	15	6	2				
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1	6			
4	15	6	2	3			
5							
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1	6			
4	15	6	2	3			
5	6	3					
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1	6			
4	15	6	2	3			
5	6	3	2				
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1	6			
4	15	6	2	3			
5	6	3	2	0			
6							

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1	6			
4	15	6	2	3			
5	6	3	2	0			
6	3	0					

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1	6			
4	15	6	2	3			
5	6	3	2	0			
6	3	0	—	—			

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1	6			
4	15	6	2	3			
5	6	3	2	0			
6	3	0	—	—			

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21			
2	78	21	3	15			
3	21	15	1	6			
4	15	6	2	3			
5	6	3	2	0			
6	3	0	–	–	3	1	0

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3		
3	21	15	1	6	3		
4	15	6	2	3	3		
5	6	3	2	0	3		
6	3	0	—	—	3	1	0

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3		
3	21	15	1	6	3		
4	15	6	2	3	3		
5	6	3	2	0	3		
6	3	0	—	—	3	1	0

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3		
3	21	15	1	6	3		
4	15	6	2	3	3		
5	6	3	2	0	3	0	
6	3	0	—	—	3	1	0

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3		
3	21	15	1	6	3		
4	15	6	2	3	3		
5	6	3	2	0	3	0	
6	3	0	–	–	3	1	0

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3		
3	21	15	1	6	3		
4	15	6	2	3	3		
5	6	3	2	0	3	0	1
6	3	0	—	—	3	1	0

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3		
3	21	15	1	6	3		
4	15	6	2	3	3		
5	6	3	2	0	3	0	1
6	3	0	—	—	3	1	0

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3		
3	21	15	1	6	3		
4	15	6	2	3	3	1	
5	6	3	2	0	3	0	1
6	3	0	—	—	3	1	0

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3		
3	21	15	1	6	3		
4	15	6	2	3	3	1	-2
5	6	3	2	0	3	0	1
6	3	0	—	—	3	1	0

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3		
3	21	15	1	6	3	-2	
4	15	6	2	3	3	1	-2
5	6	3	2	0	3	0	1
6	3	0	—	—	3	1	0

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3		
3	21	15	1	6	3	-2	3
4	15	6	2	3	3	1	-2
5	6	3	2	0	3	0	1
6	3	0	—	—	3	1	0

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3	3	
3	21	15	1	6	3	-2	3
4	15	6	2	3	3	1	-2
5	6	3	2	0	3	0	1
6	3	0	—	—	3	1	0

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3		
2	78	21	3	15	3	3	-11
3	21	15	1	6	3	-2	3
4	15	6	2	3	3	1	-2
5	6	3	2	0	3	0	1
6	3	0	—	—	3	1	0

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3	-11	
2	78	21	3	15	3	3	-11
3	21	15	1	6	3	-2	3
4	15	6	2	3	3	1	-2
5	6	3	2	0	3	0	1
6	3	0	—	—	3	1	0

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3	-11	14
2	78	21	3	15	3	3	-11
3	21	15	1	6	3	-2	3
4	15	6	2	3	3	1	-2
5	6	3	2	0	3	0	1
6	3	0	—	—	3	1	0

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Find the Bézout numbers and GCD of 99 and 78

Row	a	b	a/b	a%b	d	s	t
1	99	78	1	21	3	-11	14
2	78	21	3	15	3	3	-11
3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3
4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	-2
5	6	3	2	0	3	0	1
6	3	0	–	–	3	1	0

To check your work, verify:

$$99 * (-11) + 78 * 14 = 3$$

$$t = s_{previous} - \left(\frac{a}{b}\right) * t_{previous}$$

Bézout numbers for modular inverses

If a and m are coprime, then gcd(a, m) = 1

The extended Euclidean algorithm yields:

- 1 = gcd(a, m) = sa + tm
- So sa = 1 – tm
- Since km ≡ 0 (mod m) for any k, this means…
- sa ≡ 1 (mod m)

This means that, when a and m are coprime, the Bézout numbers reveal a's (multiplicative) inverse mod m (!)

An example

Solve the following linear congruence:

$$57x \equiv 5 \pmod{98}$$

Using the extended Euclidean algorithm on 98 and 57, we can show that 98 * (-25) + 57 * 43 = 1, so 43 is the inverse of 57 (mod 98)

Multiply by 43 on both sides

- $57x * 43 \equiv 5 * 43 \pmod{98}$
- $x \equiv 215 \pmod{98}$
- $x \equiv 19 \pmod{98}$

Solving systems of congruences

The Chinese Remainder Theorem: Let $m_1, m_2, \ldots, m_n$ be pairwise coprime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ arbitrary integers. Then the system:

- $x \equiv a_1 \pmod{m_1}$
- $x \equiv a_2 \pmod{m_2}$
- …
- $x \equiv a_n \pmod{m_n}$

has a unique solution modulo $m = m_1 m_2 \ldots m_n$

Let m be the product of the moduli, and let M_k be the product of all but the kth modulus

- Let y_k be the inverse of $M_k \pmod{m_k}$
- Now, compute $x = a_1M_1y_1 + a_2M_2y_2 + \ldots + a_nM_ny_n$

Solving systems of congruences

In *Sunzi Suanjing*, the first known example of such problems was posed:

- $x \equiv 2 \pmod 3$ $\quad m_1 = 3, a_1 = 2$
- $x \equiv 3 \pmod 5$ $\quad m_2 = 5, a_2 = 3$
- $x \equiv 2 \pmod 7$ $\quad m_3 = 7, a_3 = 2$

m = 3*5*7 = 105, and M_k is the product of all but the kth modulus

- $M_1 = 5*7 = 35$, $M_2 = 3*7 = 21$, $M_3 = 3*5 = 15$
- y_k is the inverse of $M_k \pmod{m_k}$
  - $y_1 = 2$ since $35*2 = 70 \equiv 1 \pmod 3$
  - $y_2 = 1$ since $21*1 = 21 \equiv 1 \pmod 5$
  - $y_3 = 1$ since $15*1 = 15 \equiv 1 \pmod 7$
- Then, $x = a_1M_1y_1 + a_2M_2y_2 + a_3M_3y_3$
  - $x = 2*35*2 + 3*21*1 + 2*15*1 = 233 \equiv 23 \pmod{105}$

In-class exercises

Problem 1: Find x where $8x \equiv 3 \pmod{13}$

Problem 2: Find x where:

- $x \equiv 2 \pmod{3}$
- $x \equiv 3 \pmod{5}$
- $x \equiv 4 \pmod{7}$

Fermat's Little Theorem

Theorem: If p is prime and a is an integer not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$

- This also means that $a^p \equiv a \pmod{p}$

Examples:

- Find 7^{222} **mod** 11
  - Since 11 is prime, $7^{10} \equiv 1 \pmod{11}$
  - Thus, $7^{10k} \equiv 1 \pmod{11}$ for any integer k
  - So $7^{220} \equiv 1 \pmod{11}$, and $7^{222} \equiv 7^2 \equiv 5 \pmod{11}$
  - 7^{222} **mod** 11 = 5
- Find 5^{147} **mod** 13
  - $5^{144} \equiv 1 \pmod{13}$ so $5^{147} \equiv 5^3 \equiv 8$ mod 13
  - 5^{147} **mod** 13 = 8

Primitive roots

Definition: A primitive root modulo a prime p is an integer r in $\mathbf{Z}_p$ such that every nonzero element of $\mathbf{Z}_p$ is a power of r

- We sometimes call r a generator, since multiplying r by itself repeatedly can generate every element of $\mathbf{Z}_p$
- There is a primitive root in $\mathbf{Z}_p$ for every prime p

Corollary: If b is an integer in $\mathbf{Z}_p$ and r is a primitive root modulo p, then there exists a unique exponent e in $\mathbf{Z}_p$ such that $r^e = b$

- i.e., r^e **mod** p = b
- Here, e is called the discrete log of b modulo p with base r
  - $\log_r b = e$ (where the "mod p" is understood from context)

The discrete logarithm problem

Given a prime p, a primitive root r modulo p, and a positive integer $b \in \mathbf{Z}_p$, find a value e such that
$r^e \textbf{ mod } p = b$

How would you solve this?

- No known algorithm in polynomial time

Takeaways for solving congruences:

- We can invert addition with subtraction
- We can invert multiplication with modular inverses
- Inverting exponentiation is more difficult than it appears

Final thoughts

- We can solve congruences by inverting operations, similar to standard algebra
  - To do so with multiplication, we use Euclid's algorithm and Bézout numbers to calculate multiplicative modular inverses
- The Chinese Remainder Theorem allows us to solve systems of congruences with coprime moduli
- Fermat's Little Theorem and primitive roots will come up again in cryptography
  - Section 4.5–4.6, next time!