Discrete Structures for Computer Science

William Garrison
bill@cs.pitt.edu
6311 Sennott Square

Lecture #15: Sequences and Summations

University of Pittsburgh

Today's Topics

Sequences and Summations

- Specifying and recognizing sequences
- Summation notation
- Closed forms of summations

Sequences are ordered lists of elements

Definition: A sequence is a function from a subset of the set of integers to a set S. We use the notation a_n to denote the image of the integer n. a_n is called a term of the sequence.

Examples:

- 1, 3, 5, 7, 9, 11 — A sequence with 6 terms
- 1, 1/2, 1/3, 1/4, 1/5, … — An infinite sequence

Note: The second example can be described as the sequence $\{a_n\}$ where $a_n = 1/n$ and $n \in \mathbf{Z}^+$

What makes sequences so special?

Question: Aren't sequences just sets?

Answer: The elements of a sequence are members of a set, but a sequence is ordered, a set is not.

Question: How are sequences different from ordered n-tuples?

Answer: An ordered n-tuple is ordered, but always contains n elements. Sequences can be infinite!

Some special sequences

Geometric progressions are sequences of the form $\{ar^n\}$ where a and r are real numbers

Examples:

- 1, 1/2, 1/4, 1/8, 1/16, …
- 1, -1, 1, -1, 1, -1, …

Arithmetic progressions are sequences of the form $\{a + nd\}$ where a and d are real numbers.

Examples:

- 2, 4, 6, 8, 10, …
- -10, -15, -20, -25, …

Sometimes we need to figure out the formula for a sequence given only a few terms

Questions to ask yourself:

1. Are there runs of the same value?
2. Are terms obtained by multiplying the previous value by a particular amount? (Possible geometric sequence)
3. Are terms obtained by adding a particular amount to the previous value? (Possible arithmetic sequence)
4. Are terms obtained by combining previous terms in a certain way?
5. Are there cycles amongst terms?

What are the formulas for these sequences?

Problem 1: 1, 5, 9, 13, 17, …

Problem 2: 1, 3, 9, 27, 81, …

Problem 3: 2, 3, 3, 5, 5, 5, 7, 7, 7, 7, 11, 11, 11, 11, 11, …

Problem 4: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

This is called the Fibonacci sequence.

Sequences are often specified using recurrence relations

This is a recursive approach to specifying the terms

- Later terms are specified from earlier terms

For instance, consider this definition of the Fibonacci sequence:

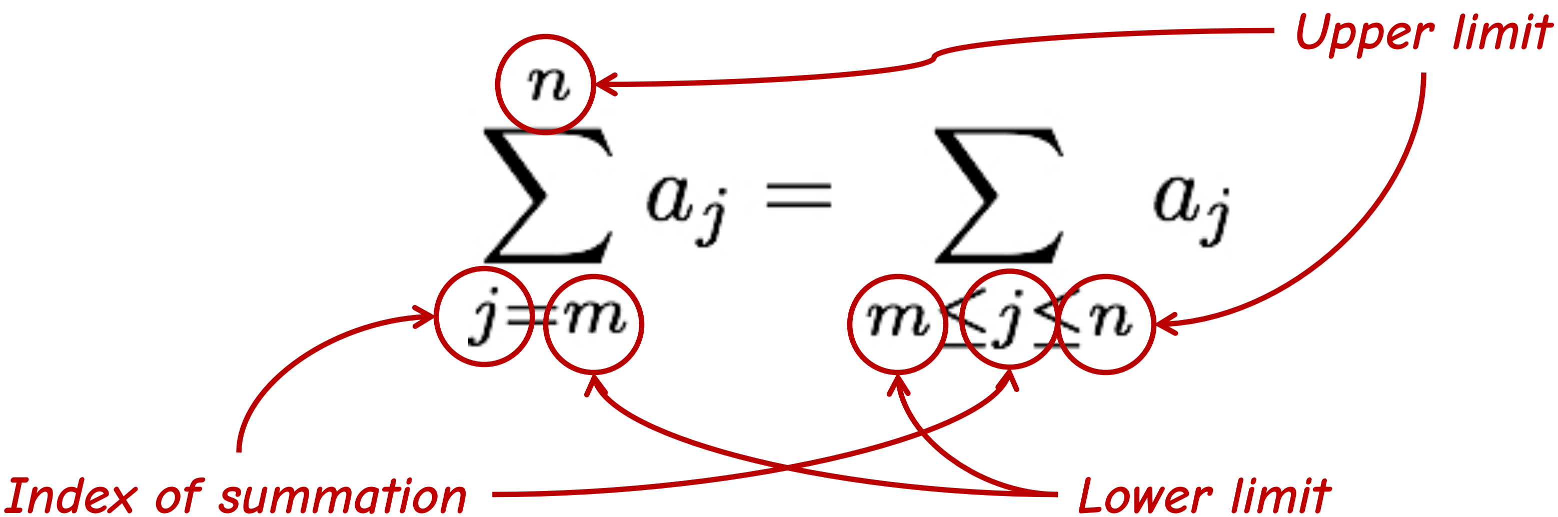
- $f_0 = 0$
- $f_1 = 1$
- For any $n > 1$, $f_n = f_{n-1} + f_{n-2}$

Note that we need at least one initial condition

- Like a base case when writing recursive code
- We'll return to recursion later in the term

Sometimes we want to find the sum of the terms in a sequence

Summation notation lets us compactly represent the sum of terms $a_m + a_{m+1} + \ldots + a_n$



$$\sum_{j=m}^{n} a_j = \sum_{m \le j \le n} a_j$$

Example: $\sum_{1 \le i \le 5} i^2 = 1 + 4 + 9 + 16 + 25 = 55$

Summations are linear: The usual laws of algebra apply

$$\sum_{j=1}^{n}(ax_j + by_j - cz_j) = a\sum_{j=1}^{n} x_j + b\sum_{j=1}^{n} y_j - c\sum_{j=1}^{n} z_j$$

Constant factors can be pulled out of the summation

A summation over a sum (or difference) can be split into a sum (or difference) of smaller summations

Example:

- $\Sigma_{1 \le j \le 3} (4j + j^2) =$
- $4\Sigma_{1 \le j \le 3}\, j + \Sigma_{1 \le j \le 3}\, j^2 =$

Example sums

Example: Express the sum of the first 50 terms of the sequence $1/n^2$ for n = 1, 2, 3, …

Answer: $$\sum_{j=1}^{50} \frac{1}{j^2}$$

Example: What is the value of $\sum_{k=4}^{8} (-1)^k$

Answer: $$\sum_{k=4}^{8} (-1)^k \quad =$$

$$=$$

$$=$$

We can also compute the summation of the elements of some set

Example: Compute $\sum_{s \in \{0,2,4,6\}} (s+2)$

Answer: (0 + 2) + (2 + 2) + (4 + 2) + (6 + 2) = 20

Example: Let f(x) = x^3 + 1. Compute $\sum_{s \in \{1,3,5,7\}} f(s)$

Answer: $f(1) + f(3) + f(5) + f(7) = 2 + 28 + 126 + 344 = 500$

Sometimes it is helpful to shift the index of a summation

This is particularly useful when combining two or more summations. For example:

$$S = \sum_{j=1}^{10} j^2 + \sum_{k=2}^{11} (2k-1)$$

Let j = k - 1

$$= \sum_{j=1}^{10} j^2 + \sum_{j=1}^{10} (2(j+1)-1)$$

Need to add 1 to each j

$$= \sum_{j=1}^{10} (j^2 + 2(j+1) - 1)$$

$$= \sum_{j=1}^{10} (j^2 + 2j + 1)$$

$$= \sum_{j=1}^{10} (j+1)^2$$

In-class exercises

On Top Hat

Summations can be nested within one another

Often, you'll see this when analyzing nested loops within a program (i.e., CS 1501/1502)

Example: Compute $\sum_{j=1}^{4}\sum_{k=1}^{3}(jk)$

Solution:

$$\begin{aligned}\sum_{j=1}^{4}\sum_{k=1}^{3}(jk) &= \sum_{j=1}^{4}(j+2j+3j)\\ &= \sum_{j=1}^{4}6j\\ &= 6+12+18+24=60\end{aligned}$$

Expand inner sum

Simplify if possible

Expand outer sum

Computing the sum of a geometric series by hand is time consuming…

Would you really want to calculate $\sum_{j=0}^{20}(6 \times 2^j)$ by hand?

Fortunately, we have a closed-form solution for computing the sum of a geometric series:

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

So, $\sum_{j=0}^{20}(6 \times 2^j) = \frac{6 \times 2^{21} - 6}{2 - 1} = 12,582,906$

Why?

Proof of geometric series closed form

There are other closed form summations that you should know

Sum	*Closed Form*
$\sum_{j=1}^{n} j$	
$\sum_{j=1}^{n} j^2$	
$\sum_{j=1}^{n} j^3$	

Final thoughts

- Sequences allow us to represent (potentially infinite) ordered lists of elements

- Summation notation is a compact representation for adding together the elements of a sequence

- Next time:
  - Midterm exam review