Discrete Structures for Computer Science

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Lecture #12: Functions

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Today's Topics

Set Functions

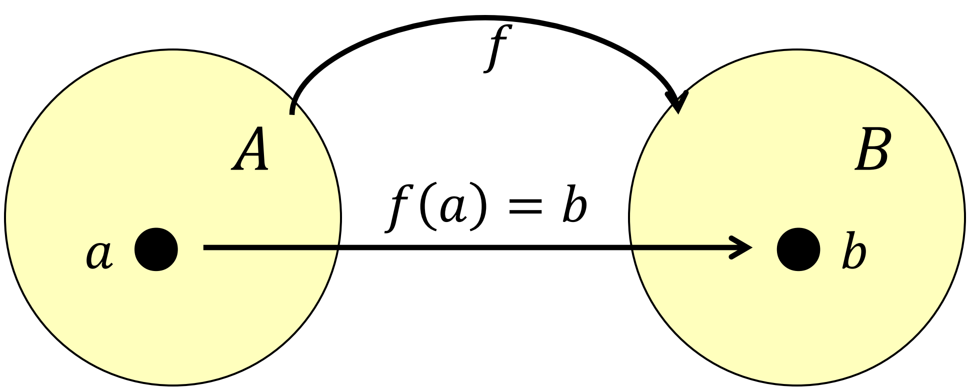
- Important definitions
- Relationships to sets, relations
- Specific functions of particular importance

Sets give us a way to formalize the concept of a function

Definition: Let A and B be nonempty sets. A function, f, from A to B is an assignment of exactly one element of set B to each element of set A.

Note: We write $f: A \rightarrow B$ to denote that f is a function from A to B

Note: We say that $f(a) = b$ if the element $a \in A$ is mapped to the unique element $b \in B$ by the function f

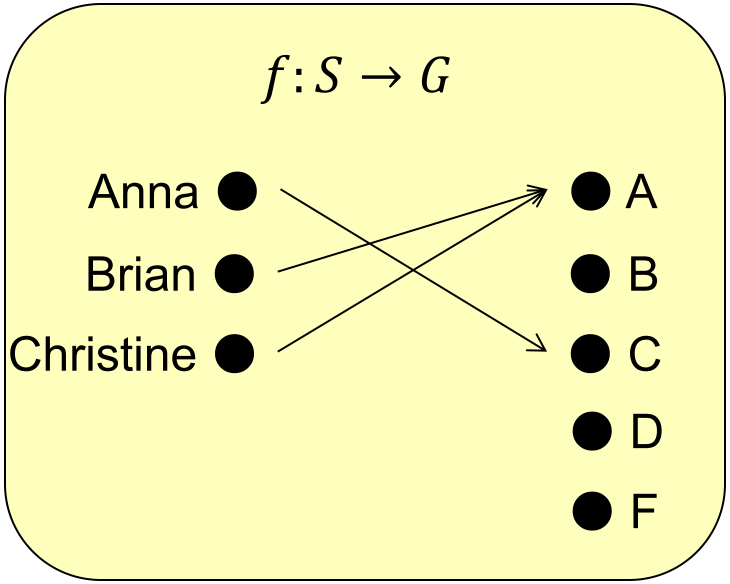


Functions can be defined in a number of ways

1. Explicitly
   - $f: \mathbf{Z} \rightarrow \mathbf{Z}$
   - $f(x) = x^2 + 2x + 1$

2. Using a programming language
   - `int min(int x, int y) = { x < y ? return x : return y; }`

3. Using a relation
   - Let S = {Anna, Brian, Christine}
   - Let G = {A, B, C, D, F}



More terminology

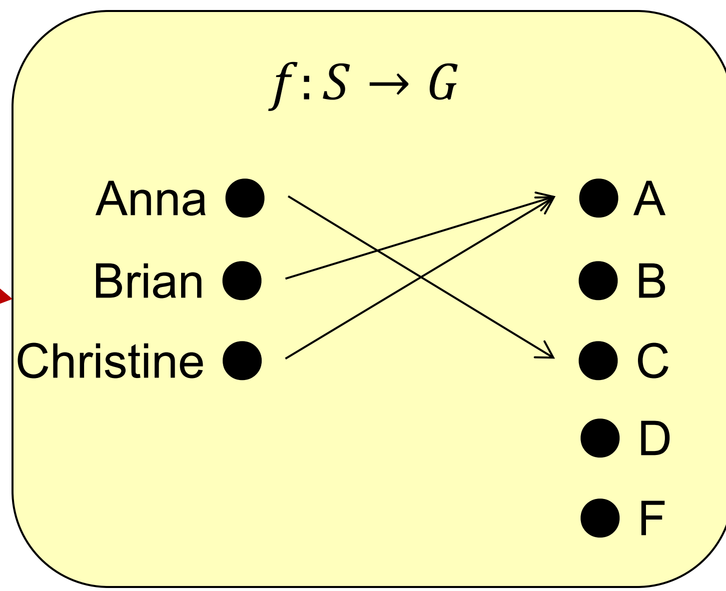
The domain of a function is the set that the function maps from, while the codomain is the set that is mapped to

If $f(a) = b$, b is called the image of a, and a is called the preimage of b

The range of a function $f: A \rightarrow B$ is the set of all images of elements of A

Domain = S = {Anna, Brian, Christine}

Codomain = G = {A, B, C, D, F}



Range = {A, C}

What are the domain, codomain, and range of the following functions?

1. $f: \mathbf{Z} \to \mathbf{Z},\ f(x) = x^3$
   - Domain:
   - Codomain:
   - Range:

2. $g: \mathbf{R} \to \mathbf{R},\ g(x) = x - 2$
   - Domain:
   - Codomain:
   - Range:

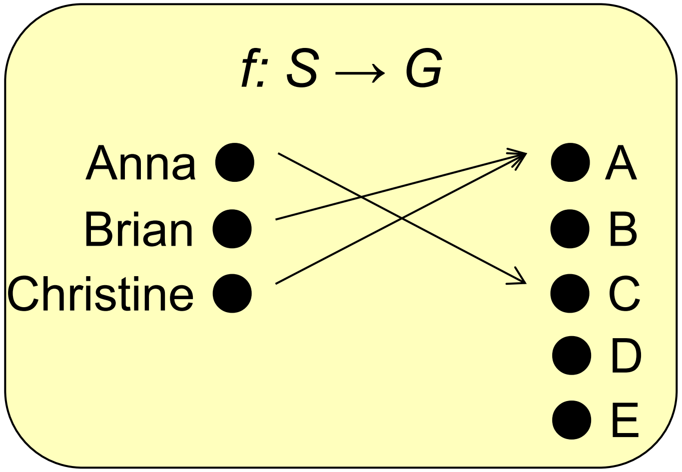
3. int foo(int x, int y) = { return (x*y)%2; }
   - Domain:
   - Codomain:
   - Range:

A one-to-one function never assigns the same image to two different elements

Definition: A function $f: A \rightarrow B$ is one-to-one, or injective, iff $\forall x, y \in A\left(\left(f(x) = f(y)\right) \rightarrow (x = y)\right)$

Are the following functions injections?

- $f: \mathbf{R} \rightarrow \mathbf{R},\ f(x) = x + 1$
- $f: \mathbf{Z} \rightarrow \mathbf{Z},\ f(x) = x^2$
- $f: \mathbf{R}^+ \rightarrow \mathbf{R}^+,\ f(x) = \sqrt{x}$
- $f: S \rightarrow G$

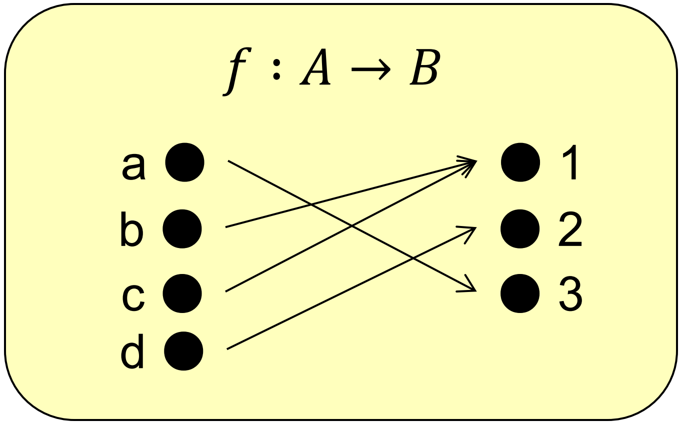


An onto function “uses” every element of its codomain

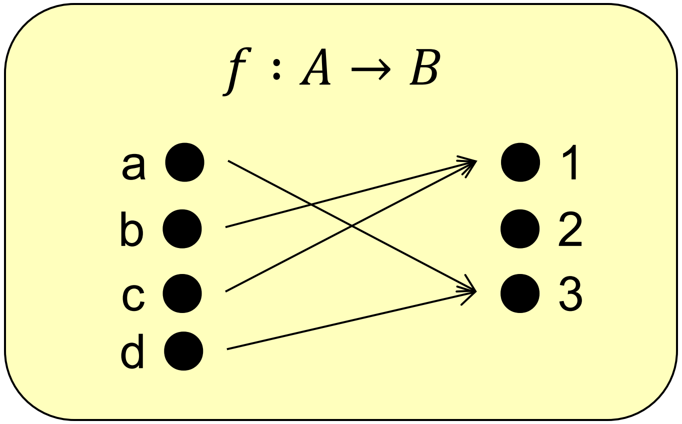
Definition: We call a function $f: A \to B$ onto, or surjective, iff $\forall b \in B\big(\exists a \in A(f(a) = b)\big)$, i.e., every element of the codomain has a preimage

Think about an onto function as “covering” the entirety of its codomain.

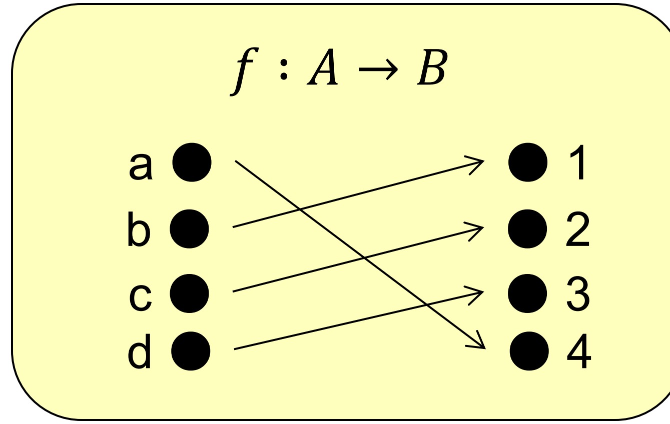
The following function is a surjection:



Are the following functions one-to-one, onto, both, or neither?

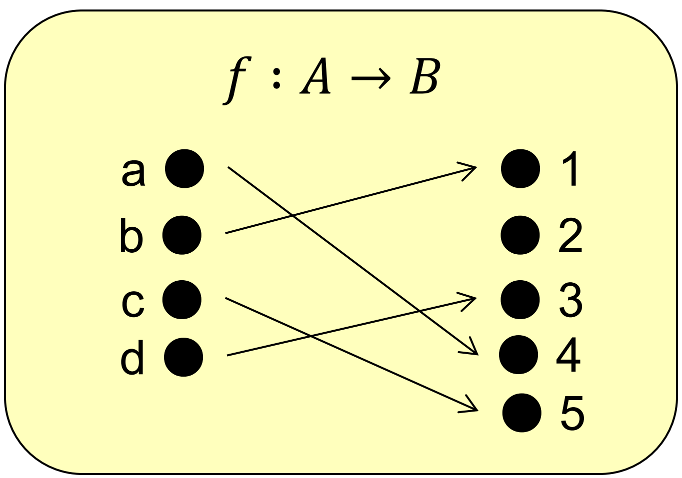


Neither!

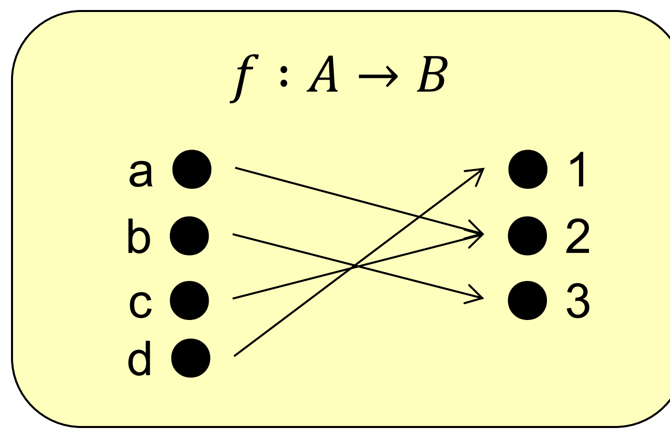


One-to-one and onto

(Aside: Functions that are both one-to-one and onto are called *bijections*)



One-to-one

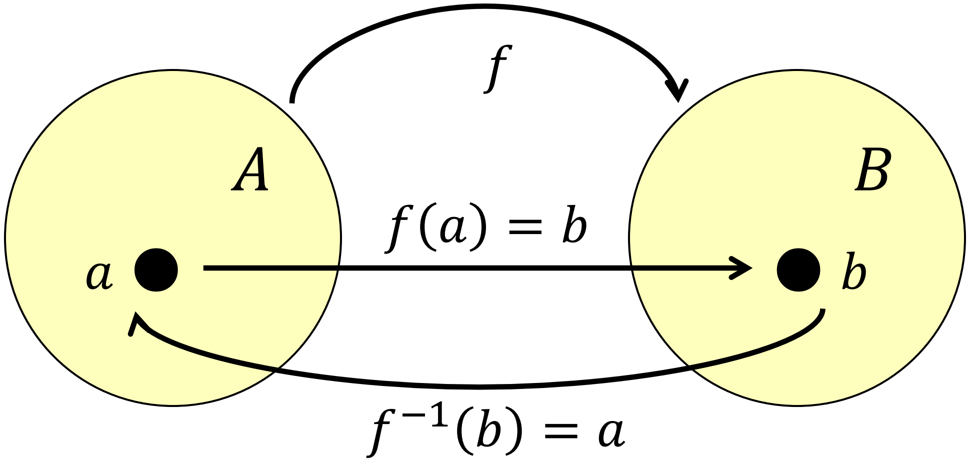


Onto

Bijections have inverses

Definition: If $f: A \rightarrow B$ is a bijection, the inverse of f is the function $f^{-1}: B \rightarrow A$ that assigns to each $b \in B$ the unique value $a \in A$ such that $f(a) = b$. That is, $f^{-1}(f(a)) = a$ and $f(f^{-1}(b)) = b$.

Graphically:



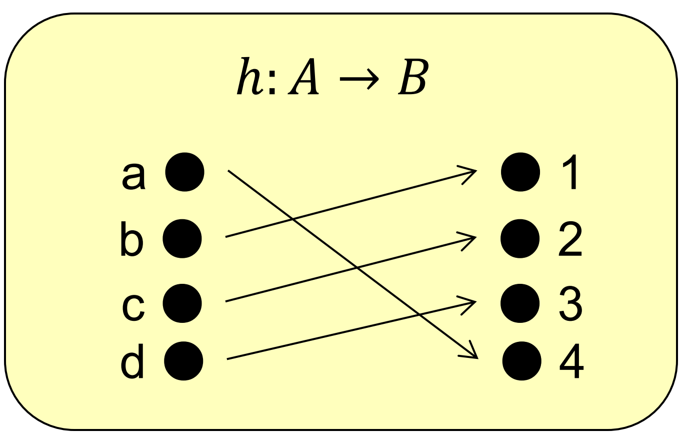
Note: Only a bijection can have an inverse. (Why?)

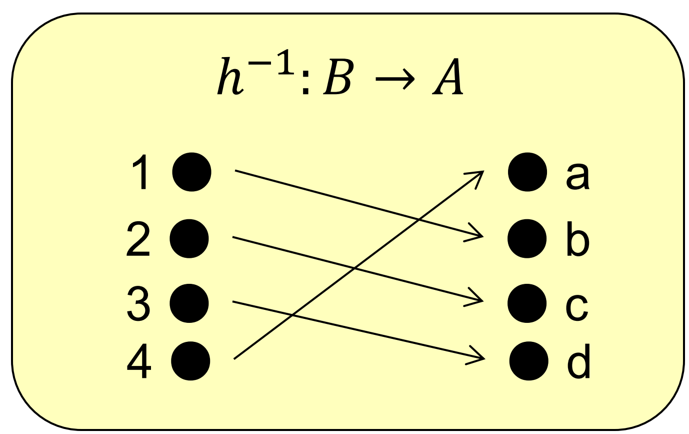
Do the following functions have inverses?

1. $f\colon \mathbf{R} \to \mathbf{R},\ f(x) = x^2$

2. $g\colon \mathbf{Z} \to \mathbf{Z},\ g(x) = x + 1$

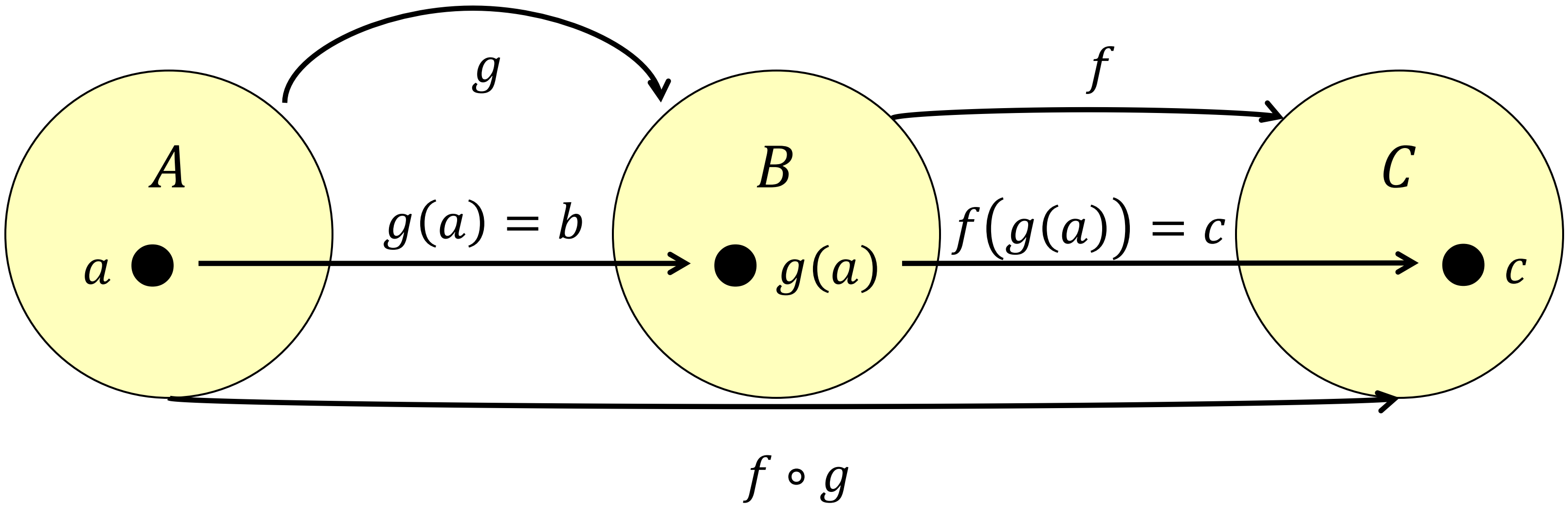
3. $h\colon A \to B$





Functions can be composed with one another

Given functions $g: A \to B$ and $f: B \to C$, the composition of f and g, denoted $f \circ g$, is defined as $(f \circ g)(x) = f\big(g(x)\big)$.



Note: For $f \circ g$ to exist, the codomain of g must be a subset of the domain of f.

Definition: If $g: A \to B$ and $f: D \to C$ and $B \subseteq D$, $f \circ g$ is a function $A \to C$ where $(f \circ g)(x) = f\big(g(x)\big)$

Can the following functions be composed? If so, what is their composition?

Let $f: A \to A$ such that $f(a) = b$, $f(b) = c$, $f(c) = a$
$g: B \to A$ such that $g(1) = b$, $g(4) = a$

1. $(f \circ g)(x)$?
2. $(g \circ f)(x)$?

Let $f: \mathbf{Z} \to \mathbf{Z}$, $f(x) = 2x + 1$
$g: \mathbf{Z} \to \mathbf{Z}$, $g(x) = x^2$

1. $(f \circ g)(x)$?
2. $(g \circ f)(x)$?

Note: There is not a guarantee that $(f \circ g)(x) = (g \circ f)(x)$.

Important functions

Definition: The floor function maps a real number x to the largest integer that is not greater than x. The floor of x is denoted $\lfloor x \rfloor$.

Definition: The ceiling function maps a real number x to the smallest integer that is not less than x. The ceiling of x is denoted $\lceil x \rceil$.

Examples:

- $\lfloor 1.2 \rfloor = 1$
- $\lfloor 7.0 \rfloor = 7$
- $\lfloor -42.24 \rfloor = -43$
- $\lceil 1.2 \rceil = 2$
- $\lceil 7.0 \rceil = 7$
- $\lceil -42.24 \rceil = -42$

We actually use floor and ceiling quite a bit in computer science…

Example: A byte, which holds 8 bits, is typically the smallest amount of memory that can be allocated on most systems. How many bytes are needed to store 123 bits of data?

Answer: We need $\lceil 123/8 \rceil = \lceil 15.375 \rceil = 16$ bytes

Example: How many 1400-byte packets can be transmitted over a 14.4 kbps modem in one minute?

Answer: A 14.4 kbps modem can transmit 14,400*60 = 864,000 bits per minute. Therefore, we can transmit $\lfloor 864000 \,/\, (1400 * 8) \rfloor = \lfloor 77.1428571 \rfloor = 77$ packets.

In-class exercises

Problem 1: Find the domain and range of each of the following functions.

a. The function that determines the number of zeros in some bit string
b. The function that maps an English word to its two rightmost letters
c. The function that assigns to an integer the sum of its individual digits

Problem 2: Suppose g is a function from A to B and f is a function from B to C. Prove that if f and g are one-to-one, then $f \circ g$ is one-to-one

Final thoughts

- Sets are the basis of functions, which are used throughout computer science and mathematics

- Next time:
  - Sequences and Summations (Section 2.4)