Lecture #9: Set Identities and Functions
Today’s Topics

Set identities

- Methods of proof
- Relationships to logical equivalences

Functions

- Important definitions
- Relationships to sets, relations
- Specific functions of particular importance
Recall from last lecture that set operations bear a striking resemblance to logical operations:

- Disjunction ($\lor$) and set union ($\cup$)
- Conjunction ($\land$) and set intersection ($\cap$)
- Negation ($\neg$) and complement ($\overline{\cdot}$)

Just as logical equivalences helped us manipulate logical expressions, **set identities** help us simplify and understand complex set definitions.
Some important set identities

<table>
<thead>
<tr>
<th>Identity</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \cup \emptyset = A$</td>
<td>Identity laws</td>
</tr>
<tr>
<td>$A \cap U = A$</td>
<td>Domination laws</td>
</tr>
<tr>
<td>$A \cup U = U$</td>
<td>Idempotent laws</td>
</tr>
<tr>
<td>$A \cap \emptyset = \emptyset$</td>
<td>Complementation law</td>
</tr>
<tr>
<td>$A \cup (B \cup C) = (A \cup B) \cup C$</td>
<td>Commutative laws</td>
</tr>
<tr>
<td>$A \cap (B \cap C) = (A \cap B) \cap C$</td>
<td>Associative laws</td>
</tr>
</tbody>
</table>

Note that set difference is **not** commutative nor associative!
Some important set identities

<table>
<thead>
<tr>
<th>Identity</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>A \cap (B \cup C) = (A \cap B) \cup (A \cap C)</td>
<td>Distributive laws</td>
</tr>
<tr>
<td>A \cup (B \cap C) = (A \cup B) \cap (A \cup C)</td>
<td>DeMorgan’s laws</td>
</tr>
<tr>
<td>A \cup (A \cap B) = A</td>
<td>Absorption laws</td>
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<tr>
<td>A \cap (A \cup B) = A</td>
<td>Complement laws</td>
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There are many ways to prove set identities

Today, we’ll discuss four common methods:

1. Membership tables
   ➢ Similar to using truth tables to prove logical equivalence.

2. Logical argument ("mutual subset" method)
   ➢ Similar to the biconditional method for proving logical equivalence.

3. Using set builder notation
   ➢ (No direct comparison to equivalences.)

4. Applying other known set identities
   ➢ Similar to using existing logical equivalences to prove new ones.
Membership tables allow us to write proofs like we did using truth tables!

The membership table for an expression has columns for sub-expressions and rows to indicate the ways in which an arbitrary element may or may not be included.

Example: A membership table for set intersection

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A ∩ B</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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</table>

An element is in A ∩ B iff it is in both A and B.
Prove that \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \)

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>B \cup C</th>
<th>A \cap (B \cup C)</th>
<th>A \cap B</th>
<th>A \cap C</th>
<th>(A \cap B) \cup (A \cap C)</th>
</tr>
</thead>
<tbody>
<tr>
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Since the appropriate columns of the membership table are the same, we can conclude that \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \). \( \square \)
Sometimes, it’s easier to make a logical argument about a set identity.

Recall: \( A = B \iff A \subseteq B \text{ and } B \subseteq A \)

As a result, we can prove a set identity by arguing that each side of the equality is a subset of the other.

**Example:** Prove that \( A \cap B = A \cup B \)

1. First prove that \( A \cap B \subseteq A \cup B \)
2. Then prove that \( A \cup B \subseteq A \cap B \)

Let’s see how this is done...

- Compare this **mutual subset** method to the biconditional method!
Prove that $A \cap B = \bar{A} \cup \bar{B}$
Prove that $A \cap B = \overline{A} \cup \overline{B}$

Since we have shown $A \cap B \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq A \cap B$, we have shown that $A \cap B = \overline{A} \cup \overline{B}$
We can use set builder notation and logical definition to make very precise proofs.

**Example:** Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

**Proof:**

1. $\overline{A \cap B} = \{x \mid x \notin A \cap B\}$  
   \hspace{1cm} \text{Def'n of complement}
2. $= \{x \mid \neg(x \in (A \cap B))\}$  
   \hspace{1cm} \text{Def'n of } \notin
3. $= \{x \mid \neg(x \in A \land x \in B)\}$  
   \hspace{1cm} \text{Def'n of } \land
4. $= \{x \mid \neg(x \in A) \lor \neg(x \in B)\}$  
   \hspace{1cm} \text{DeMorgan’s law}
5. $= \{x \mid x \notin A \lor x \notin B\}$  
   \hspace{1cm} \text{Def'n of } \notin
6. $= \{x \mid x \in \overline{A} \lor x \in \overline{B}\}$  
   \hspace{1cm} \text{Def'n of complement}
7. $= \{x \mid x \in \overline{A} \cup \overline{B}\}$  
   \hspace{1cm} \text{Def'n of } \cup
8. $= \overline{A} \cup \overline{B}$  
   \hspace{1cm} \text{Set builder notation}

Note that the argument here uses equivalence rather than subset, so we do not need to argue in both directions.
We can also construct proofs by repeatedly applying known set identities.

**Example:** Prove that \( A \cup (B \cap C) = (C \cup B) \cap \overline{A} \)

**Proof:**

1. \( A \cup (B \cap C) = A \cap (B \cap C) \) \hspace{1cm} \text{DeMorgan’s law}
2. \( = A \cap (B \cup C) \) \hspace{1cm} \text{DeMorgan’s law}
3. \( = (B \cup C) \cap \overline{A} \) \hspace{1cm} \text{Commutative law}
4. \( = (C \cup B) \cap \overline{A} \) \hspace{1cm} \text{Commutative law}

Note how similar this process is to that of proving logical equivalences using known logical equivalences. As with set builder, only one direction is needed since we’re using equivalence at every step.
In-class exercises

Problem 1: Prove DeMorgan’s law for complement over intersection using a membership table.

Problem 2: Prove the complementation law using set builder notation.
Sets give us a way to formalize the concept of a function

**Definition:** Let $A$ and $B$ be nonempty sets. A function, $f$, is an assignment of exactly one element of set $B$ to each element of set $A$.

**Note:** We write $f : A \rightarrow B$ to denote that $f$ is a function from $A$ to $B$.

**Note:** We say that $f(a) = b$ if the element $a \in A$ is mapped to the unique element $b \in B$ by the function $f$. 

![Diagram showing function from set A to set B with a mapping from a to b]
Functions can be defined in a number of ways

1. Explicitly
   - $f : \mathbb{Z} \rightarrow \mathbb{Z}$
   - $f(x) = x^2 + 2x + 1$

2. Using a programming language
   - ```
     int min(int x, int y) = { x < y ? return x : return y; }
   ```

3. Using a relation
   - Let $S = \{\text{Anna, Brian, Christine}\}$
   - Let $G = \{A, B, C, D, F\}$
More terminology

The **domain** of a function is the set that the function maps from, while the **codomain** is the set that is mapped to.

If \( f(a) = b \), b is called the **image** of a, and a is called the **preimage** of b.

The **range** of a function \( f : A \to B \) is the set of all images of elements of A.

**Domain** = \( S = \{\text{Anna, Brian, Christine}\} \)

**Codomain** = \( G = \{A, B, C, D, F\} \)

**Range** = \( \{A, C\} \)
What are the domain, codomain, and range of the following functions?

1. \( f: \mathbb{Z} \rightarrow \mathbb{Z}, \ f(x) = x^3 \)
   - Domain: \( \mathbb{Z} \)
   - Codomain: \( \mathbb{Z} \)
   - Range: Perfect cubes

2. \( g: \mathbb{R} \rightarrow \mathbb{R}, \ g(x) = x - 2 \)
   - Domain: \( \mathbb{R} \)
   - Codomain: \( \mathbb{R} \)
   - Range: \( \mathbb{R} \)

3. \( \text{int} \ foo(\text{int} \ x, \text{int} \ y) = \{ \text{return} \ (x*y) \bmod 2; \} \)
   - Domain: All \( (x, y) \in \mathbb{Z} \times \mathbb{Z} \)
   - Codomain: \( \mathbb{Z} \)
   - Range: \( \{0, 1\} \)
**Definition:** A function \( f : A \rightarrow B \) is one-to-one, or injective, iff \( \forall x,y \in A \ [ (f(x) = f(y)) \rightarrow (x = y) ] \)

Are the following functions injections?

- \( f : \mathbb{R} \rightarrow \mathbb{R}, \ f(x) = x + 1 \)
  - Yes
- \( f : \mathbb{Z} \rightarrow \mathbb{Z}, \ f(x) = x^2 \)
  - No
- \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \ f(x) = \sqrt{x} \)
  - Yes
- \( f : S \rightarrow G \)
  - No
An onto function “uses” every element of its codomain

**Definition:** We call a function $f : A \rightarrow B$ **onto**, or **surjective**, iff for every element $b \in B$, there is some element $a \in A$ such that $f(a) = b$.

Think about an onto function as “covering” the entirety of its codomain.

The following function is a **surjection**:

```
\[ f : A \rightarrow B \]

\[
\begin{array}{ccc}
a & \rightarrow & 1 \\
b & \rightarrow & 2 \\
c & \rightarrow & 3 \\
d & & \\
\end{array}
\]
Are the following functions one-to-one, onto, both, or neither?

\[ f : A \rightarrow B \]

\begin{align*}
\text{a} & \rightarrow 1 \\
\text{b} & \rightarrow 2 \\
\text{c} & \rightarrow 3 \\
\text{d} & \rightarrow 4 \\
\end{align*}

Neither!

\begin{align*}
\text{a} & \rightarrow 1 \\
\text{b} & \rightarrow 2 \\
\text{c} & \rightarrow 3 \\
\text{d} & \rightarrow 4 \\
\end{align*}

One-to-one and onto

(Aside: Functions that are both one-to-one and onto are called bijections)

\begin{align*}
\text{a} & \rightarrow 1 \\
\text{b} & \rightarrow 2 \\
\text{c} & \rightarrow 3 \\
\text{d} & \rightarrow 4 \\
\text{c} & \rightarrow 5 \\
\end{align*}

One-to-one

\begin{align*}
\text{a} & \rightarrow 1 \\
\text{b} & \rightarrow 2 \\
\text{c} & \rightarrow 3 \\
\text{d} & \rightarrow 3 \\
\end{align*}

Onto
**Definition:** If \( f : A \rightarrow B \) is a bijection, the inverse of \( f \) is the function \( f^{-1} : B \rightarrow A \) that assigns to each \( b \in B \) the unique value \( a \in A \) such that \( f(a) = b \). That is, \( f^{-1}(b) = a \) iff \( f(a) = b \).

**Graphically:**

![Diagram showing bijection and its inverse](image)

**Note:** Only a bijection can have an inverse. (Why?)
Do the following functions have inverses?

1. \( f : \mathbb{R} \rightarrow \mathbb{R}, \ f(x) = x^2 \)

No, since this function is not onto

2. \( g : \mathbb{Z} \rightarrow \mathbb{Z}, \ g(x) = x + 1 \)

Yes, \( g^{-1}(x) = x - 1 \)

3. \( h : A \rightarrow B \)

Yes

\[ h : A \rightarrow B \]

\[
\begin{align*}
a & \rightarrow 1 \\
b & \rightarrow 2 \\
c & \rightarrow 3 \\
d & \rightarrow 4 \\
\end{align*}
\]

\[ h^{-1} : B \rightarrow A \]

\[
\begin{align*}
1 & \rightarrow a \\
2 & \rightarrow b \\
3 & \rightarrow c \\
4 & \rightarrow d \\
\end{align*}
\]
Functions can be composed with one another

Given functions $g : A \rightarrow B$ and $f : B \rightarrow C$, the composition of $f$ and $g$, denoted $f \circ g$, is defined as $(f \circ g)(x) = f(g(x))$.

**Note:** For $f \circ g$ to exist, the codomain of $g$ must be a subset of the domain of $f$.

**Definition:** If $g : A \rightarrow B$ and $f : D \rightarrow C$ and $B \subseteq D$, $f \circ g$ is a function $A \rightarrow C$ where $(f \circ g)(x) = f(g(x))$.

![Diagram of function composition](image)
Can the following functions be composed? If so, what is their composition?

Let \( f : A \to A \) such that \( f(a) = b, f(b) = c, f(c) = a \)
\( g : B \to A \) such that \( g(1) = b, g(4) = a \)

1. \((f \circ g)(x)\)?
2. \((g \circ f)(x)\)?

Let \( f : \mathbb{Z} \to \mathbb{Z}, f(x) = 2x + 1 \)
\( g : \mathbb{Z} \to \mathbb{Z}, g(x) = x^2 \)

1. \((f \circ g)(x)\)?
2. \((g \circ f)(x)\)?

Note: There is not a guarantee that \((f \circ g)(x) = (g \circ f)(x)\).
Important functions

**Definition:** The floor function maps a real number \( x \) to the largest integer \( y \) that is not greater than \( x \). The floor of \( x \) is denoted \( \lfloor x \rfloor \).

**Definition:** The ceiling function maps a real number \( x \) to the smallest integer \( y \) that is not less than \( x \). The ceiling of \( x \) is denoted \( \lceil x \rceil \).

**Examples:**
- \( \lfloor 1.2 \rfloor = 1 \)
- \( \lfloor 7.0 \rfloor = 7 \)
- \( \lfloor -42.24 \rfloor = -43 \)
- \( \lceil 1.2 \rceil = 2 \)
- \( \lceil 7.0 \rceil = 7 \)
- \( \lceil -42.24 \rceil = -42 \)
We actually use floor and ceiling quite a bit in computer science...

**Example:** A byte, which holds 8 bits, is typically the smallest amount of memory that can be allocated on most systems. How many bytes are needed to store 123 bits of data?

**Answer:** We need $\lceil \frac{123}{8} \rceil = \lceil 15.375 \rceil = 16$ bytes

**Example:** How many 1400 byte packets can be transmitted over a 14.4 kbps modem in one minute?

**Answer:** A 14.4 kbps modem can transmit $14,400 \times 60 = 864,000$ bits per minute. Therefore, we can transmit $\lfloor \frac{864,000}{(1400 \times 8)} \rfloor = \lfloor 77.1428571 \rfloor = 77$ packets.
In-class exercises

Problem 3: Find the domain and range of each of the following functions.

a. The function that determines the number of zeros in some bit string
b. The function that maps an English word to its two rightmost letters
c. The function that assigns to an integer the sum of its individual digits

Problem 4: On Top Hat
Final thoughts

- Set identities are useful tools!

- We can prove set identities in a number of (equivalent) ways

- Sets are the basis of functions, which are used throughout computer science and mathematics

- Next time:
  - Summations (Section 2.4)