# Discrete Structures for Computer Science

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Lecture #8: Informal Proofs

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# Today's topics

- Proof techniques
  - How can I prove an implication is true?
  - What forms can an informal proof take?
  - How do rules of inference relate to informal proofs?

# Mathematical theorems are often stated in the form of an implication

***Example:*** If $x > y$, where $x$ and $y$ are positive real numbers, then $x^2 > y^2$.

- $\forall x, y \, [(x > 0) \wedge (y > 0) \wedge (x > y) \rightarrow (x^2 > y^2)]$
- $\forall x, y \, \big(P(x, y) \rightarrow Q(x, y)\big)$

We will discuss three applicable proof methods:

- Direct proof
- Proof by contraposition
- Proof by contradiction

# Direct proof

In a direct proof, we prove $p \rightarrow q$ by showing that if p is true, then q must necessarily be true

***Example:*** Prove that if n is an odd integer, then $n^3$ is an odd integer.

***Proof:***

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# Direct proofs are not always the easiest way to prove a given conjecture.

In this case, we can try proof by contraposition

How does this work?

- Recall that $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- Therefore, a proof of $\neg q \rightarrow \neg p$ is also a proof of $p \rightarrow q$

Proof by contraposition is an indirect proof technique since we don't prove $p \rightarrow q$ directly.

Let's take a look at an example…

# *Prove:* If n is an integer and $n^2$ is even, then n is even.

First, attempt a direct proof:

- Assume that $n^2$ is even, thus $n^2 = 2k$ for some integer $k$
- Can solve to find that $n = \sqrt{2k}$
- … ?

Now, try proof by contraposition:

- Assume $n$ is odd, thus $n = 2k + 1$ for some integer $k$
- $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$
- $= 2(2k^2 + 2k) + 1$
- Thus, $n^2$ is odd, and we proved ¬"$n$ is even" → ¬"$n^2$ is even", we can conclude that "$n^2$ is even" → "$n$ is even" □

# Proof by contradiction

Given a conditional $p \rightarrow q$, the only way to reject this claim is to prove that $p \land \neg q$ is true.

In a proof by contradiction we:

1. Assume that $p \land \neg q$ is true
2. Proceed with the proof
3. If this assumption leads us to a contradiction, we can conclude that $p \rightarrow q$ is true

# *Prove:* If n is an integer and 3n + 2 is odd, then n is odd.

***Proof:***

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**Note:** If we prove this by contraposition, we use a lot of the same mechanics/algebra!

- But the line of argumentation is different

We can also use proof by contradiction in cases where were the theorem to be proved is not of the form $p \rightarrow q$

# *Prove*: At least 10 of any 64 dates fall on the same day of the week

***Proof:***

- Let p ≡ "At least 10 of any 64 dates fall on the same day of the week"
- Assume ¬p is true, that is "64 dates can be chosen such that at most 9 fall on the same day of the week"
- When choosing dates, since there are 7 days in a week, at most $7 \times 9 = 63$ dates can be chosen in total
- This is a contradiction of the statement that we can chose 64 dates
- Therefore, the assumption is false, and we can conclude that at least 10 of any 64 dates fall on the same day of the week. □

*This proof is an example of the pigeonhole principle, which is a topic in combinatorics (the mathematical field of counting arrangements)*

# In-class exercises

**Problem 1:** Use a direct proof to show that the square of an even number is an even number.

- Note that we proved the converse earlier, so this will prove it is an "if and only if" relationship

**Problem 2:** Use proof by contraposition to show that if $n$ is an integer and $n^3 + 5$ is odd, then $n$ is even.

**Problem 3:** Prove that if $x$ is an irrational number where $x > 0$, then $\sqrt{x}$ is also irrational.

# Final Thoughts

- There are several ways to prove a statement of the form $p \to q$
  - Direct proof
  - Proof by contraposition (as we developed previously!)
  - Proof by contradiction (also useful for other forms!)

- Having several proof techniques at your disposal will make a huge difference in your success rate!

- Next lecture:
  - More proof techniques, proof strategies
  - Please read section 1.8