Discrete Structures for Computer Science

William Garrison
bill@cs.pitt.edu
6311 Sennott Square

Lecture #8: Sets

Based on materials developed by Dr. Adam Lee
Introduction to set theory

- What is a set?
- Set notation
- Basic set operations
**Definition:** A set is an unordered collection of objects.

**Examples:**
- \( A = \{1, 2, 3, 4\} \)
- \( B = \{\text{Cooper, Dougie, Mr. C}\} \)
- \( C = \{\text{motorcycle}, 3.14159, \text{Socrates}\} \)
- \( E = \{\{1, 2, 3\}, \{6, 7, 8\}, \{23, 42\}\} \)

**Informally:** Sets are really just a precise way of grouping a “bunch of stuff.”

**Sets can contain items of mixed types**

**Sets can contain other sets**
Definition: The objects making up a set are called elements of that set.

Examples:
- 3 is an element of \{1, 2, 3\}
- Azhar is an element of \{Azhar, Boipelo, Camilla, Dov\}

We can express the above examples in a more precise manner as follows:
- 3 ∈ \{1, 2, 3\}
- Azhar ∈ \{Azhar, Boipelo, Camilla, Dov\}

Question: Is 5 ∈ \{1, 2, 3, \{4, 5\}\}?
There are many different ways to describe a set

Explicit enumeration:
- \( A = \{1, 2, 3, 4\} \)

Using ellipses if the general pattern is obvious:
- \( E = \{2, 4, 6, \ldots, 98\} \)

Set builder notation (aka, set comprehensions):
- \( M = \{y \mid y = 3k \text{ for some integer } k\} \)

The set \( M \) contains...
- ... all elements \( y \)...
- ... such that...
- ... \( y = 3k \) for some integer \( k \)
There are a number of sets that are so important to mathematics that they get their own symbol:

- \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \)
- \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \)
- \( \mathbb{Z}^+ = \{1, 2, \ldots\} \)
- \( \mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\} \)
- \( \mathbb{R} \)
- \( \emptyset = \{\} \)

**Note:** This notation differs from book to book

- Some authors write these sets as \( \mathbb{N}, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Q}, \) and \( \mathbb{R} \)
  - I’ll do so in handwriting (“blackboard bold”)
- Some authors do not include zero in the natural numbers
  - I like the above because it makes \( \mathbb{N} \neq \mathbb{Z}^+ \) (more expressive)

Be careful when reading other books or researching on the Web, as things may be slightly different!
You’ve actually been using sets implicitly all along!

**Mathematics**

**Function** min(int x, int y) : int

```
if x < y then
    return x
else
    return y
endif
end function
```

**Domains of propositional functions**

\[ F(x,y) \equiv x \text{ and } y \text{ are friends} \]

Domain: “All people”

\[ \forall x \exists y F(x,y) \]

**Programming language data types**
**Set equality**

*Definition:* Two sets are *equal* if and only if they contain exactly the same elements.

*Mathematically:* $A = B$ iff $\forall x \ (x \in A \iff x \in B)$

*Example:* Are the following sets equal?

- {1, 2, 3, 4} and {1, 2, 3, 4}  
  *Yes!*
- {1, 2, 3, 4} and {4, 1, 3, 2}  
  *Yes!*
- {a, b, c, d, e} and {a, a, c, b, e, d}  
  *Yes!*
- {a, e, i, o} and {a, e, i, o, u}  
  *No!*
We can use Venn diagrams to graphically represent sets.

The set $V$ of all vowels is contained within the universe of “all letters.”

Sometimes, we add points for the elements of a set.
Sets can be contained within one another

**Definition:** Some set $A$ is a **subset** of another set $B$ iff every element of $A$ is an element in the set $B$. We denote this fact as $A \subseteq B$, and call $B$ a **superset** of $A$.

**Graphically:**

```
  B
  / \                          / \  U
 /   \        A
```

**Mathematically:**

**Definition:** We say that $A$ is a **proper subset** of $B$ iff $A \subseteq B$, but $A \neq B$. We denote this by $A \subset B$. More precisely:
Properties of subsets

Property 1: For all sets $S$, we have that $\emptyset \subseteq S$

Proof: The set $\emptyset$ contains no elements. So, trivially, every element of the set $\emptyset$ is contained in any other set $S$. □

Property 2: For any set $S$, $S \subseteq S$.

Property 3: If $S_1 = S_2$, then $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$. 
Note: Differences between $\subseteq$ and $\in$

Recall that $A \subseteq B$ if $A$ is a **subset** of $B$, whereas $a \in A$ means that $a$ is an **element** of $A$.

**Examples:**

- Is $\{1\} \in \{1, 2, 3\}$?
- Is $\{1\} \subseteq \{1, 2, 3\}$?
- Is $1 \in \{1, 2, 3\}$?
- Is $\{2, 3\} \subseteq \{1, \{2, 3\}, \{4, 5\}\}$?
- Is $\{2, 3\} \in \{1, \{2, 3\}, \{4, 5\}\}$?
- Is $\emptyset \in \{1, 2, 3\}$?
- Is $\emptyset \subseteq \{1, 2, 3\}$?
Problem 1: Come up with two ways to represent each of the following sets:

- The even integers
- Negative integers between -1 and -10, inclusive
- The positive integers

Problem 2: Draw a Venn diagram representing the sets \( \{1, 2, 3\} \) and \( \{3, 4, 5\} \).

Problem 3-4: On Top Hat
We can create a new set by combining two or more existing sets

**Definition:** The union of two sets $A$ and $B$ contains every element that is either in $A$ or in $B$. We denote the union of the sets $A$ and $B$ as $A \cup B$.

**Graphically:**

![Venn Diagram](image)

**Mathematically:**

$$A \cup B = \{ x | x \in A \lor x \in B \}$$

**Example:** $\{1, 2, 3\} \cup \{6, 7, 8\} = \{1, 2, 3, 6, 7, 8\}$
We can take the union of any number of sets.

**Example:** \( A \cup B \cup C \)

**Graphically:**

In general, we can express the union \( S_1 \cup S_2 \cup ... \cup S_n \) using the following notation:

\[
\bigcup_{i=1}^{n} S_i
\]

This is just like summation notation!
Sometimes we’re interested in the elements that are in more than one set

**Definition:** The intersection of two sets A and B contains every element that is in A and also in B. We denote the intersection of the sets A and B as \( A \cap B \).

**Graphically:**

![Venn Diagram](https://via.placeholder.com/150)

**Mathematically:**

\[
A \cap B = \{x \mid x \in A \land x \in B\}
\]

**Examples:**

- \( \{1, 2, 3, 7, 8\} \cap \{6, 7, 8\} = \{7, 8\} \)
- \( \{1, 2, 3\} \cap \{6, 7, 8\} = \emptyset \)

We say that two sets A and B are **disjoint** if \( A \cap B = \emptyset \)
We can take the intersection of any number of sets

Example: \( A \cap B \cap C \)

Graphically:

As with the union operation, we can express the intersection \( S_1 \cap S_2 \cap \ldots \cap S_n \) as:

\[
\bigcap_{i=1}^{n} S_i
\]
Set differences

**Definition:** The *difference* of two sets A and B, denoted by \( A - B \), contains every element that is in A, but not in B.

**Graphically:**

![Venn diagram showing set difference](image)

**Mathematically:**

\[
A - B = \{ x | x \in A \land x \notin B \}
\]

**Example:** \( \{1, 2, 3, 4, 5\} - \{4, 5, 6, 7, 8\} = \{1, 2, 3\} \)

**Be careful:** Some authors use the notation \( A \setminus B \) to denote the set difference \( A - B \).
If we have specified a universe $U$, we can determine the complement of a set $A$.

**Definition:** The complement of a set $A$, denoted by $\overline{A}$, contains every element that is in $U$, but not in $A$.

**Graphically:**

![Complement Diagram]

**Mathematically:**

$$A = \{x \mid x \in U \land x \notin A\}$$

**Examples:** Assume that $U = \{1, 2, \ldots, 10\}$

- $\{1, 2, 3, 4, 5\} = \{6, 7, 8, 9, 10\}$
- $\{2, 4, 6, 8, 10\} = \{1, 3, 5, 7, 9\}$
Definition: Let $S$ be a set. If there are exactly $n$ elements in $S$, where $n$ is a nonnegative integer, then $S$ is a finite set whose cardinality is $n$. The cardinality of $S$ is denoted by $|S|$.

Example: If $S = \{a, e, i, o, u\}$, then $|S| =$

Useful facts: If $A$ and $B$ are finite sets, then

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A - B| = |A| - |A \cap B|$

Aside: We’ll talk about the cardinality of infinite sets later in the course.
Definition: Given a set $S$, its **power set** is the set containing all subsets of $S$. We denote the power set of $S$ as $P(S)$.

Examples:

- $P(\{1\}) = \{\emptyset, \{1\}\}$
- $P(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2, 3\}, \{1, 2, 3\}\}$

Note:

- The set $\emptyset$ is in the power set of any set $S$
- The set $S$ is in its own power set
- $|P(S)| = 2^{|S|}$
- Some authors use the notation $2^S$ to represent the power set of $S$
Be careful when computing power sets

**Question:** What is \( P\{1, 2, \{1, 2\}\} \)?

**Note:** The set \( \{1, 2, \{1, 2\}\} \) has three elements

- 1
- 2
- \( \{1, 2\} \)

So, we need all combinations of those elements:

- \( \emptyset \)
- \( \{1\} \)
- \( \{2\} \)
- \( \{\{1,2\}\} \)
- \( \{1, 2\} \)
- \( \{1, \{1,2\}\} \)
- \( \{2, \{1,2\}\} \)
- \( \{1, 2, \{1,2\}\} \)

\[ \therefore P(\{1, 2, \{1,2\}\}) = \{\emptyset, \{1\}, \{2\}, \{\{1,2\}\}, \{1, 2\}, \{1, \{1,2\}\}, \{2, \{1,2\}\}, \{1, 2, \{1,2\}\}\} \]

*This power set has \( 2^3 = 8 \) elements.*
How do we represent ordered collections?

**Definition:** The ordered n-tuple \((a_1, a_2, \ldots, a_n)\) is the ordered collection that has \(a_1\) as its first element, \(a_2\) as its second element, \(\ldots\), and \(a_n\) as its \(n^{th}\) element.

**Note:** \((a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)\) iff \(a_i = b_i\) for \(i = 1, \ldots, n\).

**Special case:** Ordered pairs of the form \((x \in \mathbb{Z}, y \in \mathbb{Z})\) are the basis of the Cartesian plane!

- \((a, b) = (c, d)\) iff \(a = c\) and \(b = d\)
- \((a, b) = (b, a)\) iff \(a = b\)

How can we construct and describe ordered n-tuples?
We use the Cartesian product operator to construct ordered n-tuples

**Definition:** If A and B are sets, the **Cartesian product** of A and B, which is denoted $A \times B$, is the set of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$.

**Mathematically:**

**Examples:** Let $A = \{1, 2\}$ and $B = \{y, z\}$

- What is $A \times B$?
- $B \times A$?
- Are $A \times B$ and $B \times A$ equivalent?
Cartesian products can be made from more than two sets

**Example:** Let

- $S = \{x \mid x \text{ is enrolled in CS 441}\}$
- $G = \{x \mid x \in \mathbb{R} \land 0 \leq x \leq 100\}$
- $Y = \{\text{freshman, sophomore, junior, senior}\}$

The set $S \times Y \times G$ consists of all possible (CS441 student, year, grade) combinations.

**Note:** My grades database is a subset of $S \times Y \times G$ that defines a relation between students in the class, their year at Pitt, and their grade!

*We will study the properties of relations towards the end of this course.*
Sets and Cartesian products can be used to represent trees and graphs

Let:

- \( N = \text{All names} \)
- \( F = N \times N \)

A social network can be represented as a graph \((V, E)\) in which the set \(V\) denotes the people in the network and the set \(E\) denotes the set of “friendship” links: \((V, E) \in P(N) \times P(F)\)

In the above network:

- \( V = \{\text{Avetis, Bluma, ..., Colin}\} \subseteq N \)
- \( E = \{(\text{Avetis, Bluma}), (\text{Avetis, Duru}), ..., (\text{Sekai, Colin})\} \subseteq N \times N \)
Set notation allows us to make quantified statements more precise.

We can use set notation to make the domain of a quantified statement explicit.

**Example:** \( \forall x \in \mathbb{R} \ (x^2 \geq 0) \)
- The square of any real number is at least zero.

**Example:** \( \forall n \in \mathbb{Z} \ \exists j,k \in \mathbb{Z} \ [(3n+2 = 2j+1) \rightarrow (n = 2k+1)] \)
- If \( n \) is an integer and \( 3n + 2 \) is odd, then \( n \) is odd.

**Note:** This notation is far less ambiguous than simply stating the domains of propositional functions. In the remainder of the course, we will use this notation whenever possible.
Definition: Given a predicate \( P \) and its corresponding domain \( D \) the truth set of \( P \) enumerates all elements in \( D \) that make the predicate \( P \) true.

Examples: What are the truth sets of the following predicates, given that their domain is the set \( \mathbb{Z} \)?

- \( P(x) \equiv |x| = 1 \)
- \( Q(x) \equiv x^2 > 0 \)
- \( R(x) \equiv x^5 = 1049 \)

Note:

- \( \forall x \ P(x) \) is true iff the truth set of \( P \) is the entire domain \( D \)
- \( \exists x \ P(x) \) is true iff the truth set of \( P \) is non-empty
How do computers represent and manipulate finite sets?

**Observation:** Representing sets as unordered collections of elements (e.g., arrays of Java Object data types) can be inefficient.

As a result, sets are usually represented using either hash maps or bitmaps.

You’ll learn about these in 1501, so today we’ll focus on bitmap representations.

This is probably best explained through an example...
Playing with the set \( S = \{x \mid x \in \mathbb{N}, x < 10\} \)

To represent a set as a bitmap, we must first agree on an ordering for the set. In the case of \( S \), let’s use the natural ordering of the numbers.

Now, any subset of \( S \) can be represented using \(|S| = 10\) bits. For example:

- \( \{1, 3, 5, 7, 9\} = 0101 \ 0101 \ 01 \)
- \( \{1, 1, 1, 4, 5\} = 0100 \ 1100 \ 00 \)

What subsets of \( S \) do the following bitmaps represent?

- \( 0101 \ 1010 \ 11 \)
- \( 1111 \ 0000 \ 10 \)
Set operations can be carried out very efficiently as bitwise operations.

**Example:** \{1, 3, 7\} ∪ \{2, 3, 8\}

\[
0101\ 0001\ 00 \\
\lor\ 0011\ 0000\ 10 \\
\underline{0011\ 0000\ 10}
\]

**Example:** \{1, 3, 7\} ∩ \{2, 3, 8\}

\[
0101\ 0001\ 00 \\
\land\ 0011\ 0000\ 10 \\
\underline{0001\ 0000\ 00}
\]

**Note:** These operations are much faster than searching through unordered lists!
Set operations can be carried out very efficiently as bitwise operations.

**Example:** \( \{1, 3, 7\} \)

\[
\begin{array}{c}
\text{\small \( \neg (0101 0001 00) \)} \\
\text{\small \( 1010 1110 11 \)} \\
\hline
\text{\small \( 1010 1110 11 \)} = \{0, 2, 4, 5, 6, 8, 9\}
\end{array}
\]

Since the set difference \( A - B \) can be written as \( A \cap (A \cap B) \), we can calculate it as \( A \land \neg (A \land B) \).

Although set difference is more complicated than the basic operations, it is still much faster to calculate set differences using a bitmap approach as opposed to an unordered search.
Problem 4: Let $A = \{1, 2, 3, 4\}$, $B = \{3, 5, 7, 9\}$, and $C = \{7, 8, 9, 10\}$. Calculate the following:
- $A \cap B$
- $A \cup B \cup C$
- $B \cap C$
- $A \cap B \cap C$

Problem 5: Come up with a bitmap representation of the sets $A = \{a, c, d, f\}$ and $B = \{a, b, c\}$. Use this to calculate the following:
- $A \cup B$
- $A \cap B$
Final thoughts

- Sets are one of the most basic data structures used in computer science.

- Today, we looked at:
  - How to define sets
  - Basic set operations
  - How computers represent sets

- Next time:
  - Set identities (Section 2.2)
  - Functions (Section 2.3)